Motion planning of underactuated mechanical systems: oscillatory control and kinematic controllability

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Outline

1 Affine connection control systems

Simple (constrained)* mechanical systems

(i) n-dimensional configuration space Q.

(ii) Riemannian metric $\mathcal{G}$ on Q.

(iii) Potential energy $V \in C^\infty(Q)$.

(iv) $\mathcal{F} = \{F^1, \dots, F^m\}$, co-vector fields on Q (the controls).

(v) $k : TQ \to TQ$ damping tensor.

(vi) * Constraints $\mathcal{D} : Q \to TQ$.

$$m < n \quad \Longrightarrow \quad \text{Underactuated system}$$
$$m = n \quad \Longrightarrow \quad \text{Fully actuated system}$$

Associated with $\mathcal{G} \Rightarrow$ the Levi-Civita affine connection

$$\nabla^{\mathcal{G}}_X Y = \left(X^b \frac{\partial Y^a}{\partial q^b} + \underbrace{\Gamma^a_{bc}}_{\text{Christoffel symbols}} X^b Y^c \right) \frac{\partial}{\partial q^a}$$

$$\Gamma^a_{bc} = \nabla^{\mathcal{G}}_{\frac{\partial}{\partial q^b}} \frac{\partial}{\partial q^c} = \frac{1}{2} \mathcal{G}^{ad} \left(\frac{\partial \mathcal{G}_{db}}{\partial q^c} + \frac{\partial \mathcal{G}_{dc}}{\partial q^b} - \frac{\partial \mathcal{G}_{bc}}{\partial q^d} \right)$$

Euler-Lagrange equations for $L(q,\dot q) = \frac{1}{2}\mathcal{G}(\dot q,\dot q)$ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0 \Leftrightarrow \ddot q + \Gamma_{ab}(q)\dot q^a \dot q^b = 0 \Leftrightarrow \underbrace{\boxed{\nabla^{\mathcal{G}}_{\dot q}\dot q = 0}}_{\text{geodesics of } \mathcal{G}}$$

Geodesic spray: vector field on TQ whose integral curves are the geodesics,

$$Z = v^a \frac{\partial}{\partial q^a} - \Gamma^a_{bc} v^b v^c \frac{\partial}{\partial v^a}$$

Geodesics satisfy $\dot x = Z(x)$, $x = (q,v) \in TQ$.

The dynamics of the mechanical system is described by the forced Euler-Lagrange equations for the Lagrangian $L = \frac{1}{2}\mathcal{G} - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = \mathcal{G}k(\dot q) + \sum_{i=1}^m u_i(t) F^i \,,$$

⬇

$$\ddot q^c + \Gamma^c_{ab}(q)\dot q^a \dot q^b = -g^{ac}\frac{\partial V}{\partial q^a} + k^c_a \dot q^a + \sum_{i=1}^m u_i Y_i^a \,, \quad Y_i = \mathcal{G}^{-1}F^i$$

On TQ

$\dot x = Z - (\operatorname{grad} V)^v + k^v + Y^v$

$Y^v = \sum_i u_i Y_i^a \frac{\partial}{\partial v^a}$ vert. lift

⇄

On Q

$\nabla^{\mathcal{G}}_{\dot q}\dot q = -\operatorname{grad} V + k(\dot q) + Y$

Observations:

- Intrinsic formulation
- Second-order equations on Q
- Inputs as "accelerations"

Nonholonomic systems: $TQ = \mathcal{D} \oplus \mathcal{D}^{\perp}$

$$\mathcal{P} : TQ \to \mathcal{D}\,, \quad \mathcal{Q} : TQ \to \mathcal{D}^{\perp}$$

Nonholonomic affine connection: $\overline{\nabla}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X \mathcal{Q})(Y)$. [Synge, 1928]

Forced Lagrange-d'Alembert equations,

$$\overline{\nabla}_{\dot{q}(t)}\dot{q}(t) = -\mathcal{P}(\operatorname{grad} V) + \mathcal{P}(k(\dot{q})) + \textstyle\sum_{i=1}^{m} u_i(t)\mathcal{P}(Y_i(q(t)))\,,\ \dot{q}(0) \in \mathcal{D}.$$

Also admits expression of the form $\dot{x} = f(x) + \sum_i u_i(t) g_i(x)$.

Thus we focus on affine connection systems,

$$\nabla_{\dot{q}(t)}\dot{q}(t) = Y_0(q) + k(\dot{q}) + \sum_{i=1}^{m} u_i(t) Y_i(q(t))\,.$$

Symmetric product: $X, Y \in \mathfrak{X}(Q) \Longrightarrow \langle X : Y \rangle \equiv \nabla_X Y + \nabla_Y X$.

[$\sim$ Beltrami bracket of functions: gradient control systems, Crouch, 1981; Van der Schaft, 1984]

Geometric meaning! [Lewis, 1998]

A distribution $\mathcal{D}$ on Q is closed under the symmetric product $\iff$ $\mathcal{D}$ is geodesically invariant

(Complementary to the Lie bracket case)

Tools for motion planning: analysis

Configuration accessibility & controllability [Lewis & Murray, 1997; Cortés, Martínez, Bullo, 2001]

Accelerations at disposal $\{Y_a\}$

$\Downarrow$

Accessible velocities $\{Y_a, \langle Y_b : Y_c\rangle, \dots\}$

$\Downarrow$

Accessible configurations $\{Y_a, \langle Y_b : Y_c\rangle, [Y_b, Y_c], [\langle Y_a : Y_b\rangle, Y_c], \dots\}$

Series expansions describing the evolution of the trajectories [Bullo, 2001; Cortés, Martínez 2000; Martínez, Cortés, 2001]

Tools for motion planning: design

Motion planning via perturbation analysis: systems evolving on Lie groups [Leonard & Krishnaprasad, 1995], undulatory locomotion systems [Ostrowski, 2000].

Motion planning via series expansions: systems evolving on Lie groups [Bullo, Leonard, Lewis, 2000], systems evolving on principal fiber bundles [Martínez, Cortés, 2001].

Here we focus on motion planning via

- oscillatory control
- Motion planning via kinematic controllability

2 Perturbation analysis for oscillatory control

F. Bullo, Siam J Ctrl & Optm '00 (To appear), S. Martínez, J. Cortés, F. Bullo, TAC '01 (Submitted)

General setting:

$$\frac{dx}{dt} = f(x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, x\right), \qquad x(0) = x_0\,.$$

Define the pull-back vector field F

$$F(t,x) = \left(\left(\Phi_{0,t}^{g(t,x)}\right)^{*} f\right)(x)\,.$$

Theorem

If the vector fields in $\{g_\tau,\ \tau \in [0,T]\}$ commute and are T-periodic and zero mean $(g(t+T,x) = g(t,x)$ and $\int_0^T g(t,x)dt = 0)$ ⟹ F is T-periodic.

Consider then

$$\overline{F}(x) = \frac{1}{T}\int_0^T F(t,x)dt\,.$$

and

$$\frac{dz}{dt} = F\left(\frac{t}{\epsilon}, z\right), \quad z(0) = x_0,$$

$$\frac{dy}{dt} = \overline{F}(y), \quad y(0) = x_0.$$

Theorem

$$x(t) = \Phi_{0,t/\epsilon}^{g(t,x)}(z(t))\,, \quad t \in \mathbb{R}_+$$

$z(t) - y(t) = O(\epsilon)$, as $\epsilon \to 0$ on the time scale 1

2.1 Single & multiple input cases

$$g\left(\frac{t}{\epsilon}, x\right) = u\left(\frac{t}{\epsilon}\right) g(x)\,,\; U_k(t) = \frac{1}{k!}\left(\int_0^t u(\tau)d\tau\right)^k,\; \overline{U}_k = \frac{1}{T}\int_0^T U_k(t)dt.$$

Theorem

Let $t \mapsto u(t)$ be a bounded function, T-periodic and zero-mean,

$$F(t,x) = f(x) + \sum_{k=1}^{+\infty} U_k(t)\,\mathrm{ad}_g^k\, f(x)$$

and its average $\overline{F}$ satisfies

$$\overline{F}(x) = f(x) + \sum_{k=1}^{+\infty} \overline{U}_k\,\mathrm{ad}_g^k\, f(x)\,.$$

The result can be extended to the multi-input case.

All the previous discussion can also be extended for two-time scale setting:

$$\frac{dx}{dt} = f(t,x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, t, x\right), \qquad x(0) = x_0\,.$$

This is very useful, since in the averaged system we still have explicit dependence on t. This allows us to control the averaged system.

2.2 Mechanical control systems

$$\nabla_{\dot q}\dot q = \text{Gravity} + \text{Damping} \;+\; \frac{1}{\epsilon}\sum_i u_i\left(\frac{t}{\epsilon}\right) Y_i(q)$$

$u_i(t)$ is T-periodic and zero-mean

$$\nabla_{\dot r}\dot r = \text{Gravity} + \text{Damping} \;+\; \sum_{i=1}^{m}\left(\frac{1}{2}\overline{U}_{\{i\}}^2 - \overline{U}_{\{i,i\}}\right)\langle Y_i : Y_i\rangle(r)$$
$$+\sum_{i<j}\left(\overline{U}_{\{i\}}\overline{U}_{\{j\}} - \overline{U}_{\{i,j\}} - \overline{U}_{\{j,i\}}\right)\langle Y_i : Y_j\rangle(r)$$

as $\epsilon \longrightarrow 0$ on appropriate time scale

Motion planning strategies 2. Oscillatory control

Parenthesis on second perturbation setting

$$H(q,p,v(t)) = \frac{1}{2}p' M(q)^{-1}p \;+\; V(q) \;+\; \frac{1}{\epsilon}v\left(\frac{t}{\epsilon}\right)\varphi(q)$$

$$H_{\text{averaged}}(q,p) = \frac{1}{2}p' M(q)^{-1}p \;+\; V(q) + \lambda\langle\varphi:\varphi\rangle(q)$$

Beltrami bracket

$$\langle\varphi:\varphi\rangle = \frac{\partial\varphi}{\partial q}' M^{-1}\frac{\partial\varphi}{\partial q}$$

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Two-time scales analysis

$$\nabla_{\dot{q}}\dot{q} = \text{Gravity} + \text{Damping} + \sum_i \left(v_i(t) + \frac{1}{\epsilon} u_i\left(\frac{t}{\epsilon}, t\right)\right) Y_i(q)$$

$u_i(\tau, t)$ is T-periodic and zero-mean in τ

⬇

$$\nabla_{\dot{r}}\dot{r} = \text{Grav} + \text{Damp} + \sum_i v_i(t) Y_i(q) + \sum_{i=1}^{m} \left(\frac{1}{2}\overline{U}^2_{\{i\}}(t) - \overline{U}_{\{i,i\}}(t) \right)$$

$$\langle Y_i : Y_i \rangle (r) + \sum_{i<j} \left(\overline{U}_{\{i\}}(t)\overline{U}_{\{j\}}(t) - \overline{U}_{\{i,j\}}(t) - \overline{U}_{\{j,i\}}(t) \right) \langle Y_i : Y_j \rangle (r)$$

Tracking: Problem statement

Given a smooth desired curve $q^d : [0, T] \to Q$, $q^d(0) = q(0)$, $\dot{q}^d(0) = \dot{q}(0)$, find controls laws $w_i : Q \times [0, T] \to \mathbb{R}^m$ such that $q(t)$ approximates $q^d(t)$ up to an error of order ϵ.

Controllability assumption:

- $\text{span}\{Y_i(q), \langle Y_i : Y_j \rangle (q) \mid 1 \leq i \leq m, 1 \leq i < j \leq m\}$ has maximal rank for every $q \in Q$
- Every bad symmetric product $\langle Y_i : Y_i \rangle (q)$ can be put as a linear combination of the input vector fields $\{Y_i(q)\}_{i=1}^m$.

Accordingly,

(i) there exist functions $z_i^d, z_{jk}^d : [0,T] \to \mathbb{R}$, for $j < k$, such that

$$\nabla_{\dot{q}^d}\dot{q}^d = g_0(q^d) + g_1(q^d)\dot{q}^d + \sum_i z_i^d(t) Y_i(q^d) + \sum_{j<k} z_{jk}^d(t) \langle Y_j : Y_k \rangle (q^d),$$

(ii) there exist smooth functions $\alpha_{ij} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\langle Y_i : Y_i \rangle (q) = \sum_j \alpha_{ij}(q) Y_j(q), \qquad \forall q \in Q.$$

Enumerate the pairs $1 \le a < b \le m$ by $(a,b) \mapsto N(a,b) \in \{1, \ldots, N\}$.

$$\psi_{N(a,b)}(t) = \sqrt{2}\, N(a,b) \cos(N(a,b)\, t).$$

Given $q^d : [0,T] \to Q$, $q^d(0) = q(0)$, $\dot{q}^d(0) = \dot{q}(0)$. The trajectory $q(t)$ of the mechanical control system equals q^d up to order ϵ over the time scale 1 with the controls

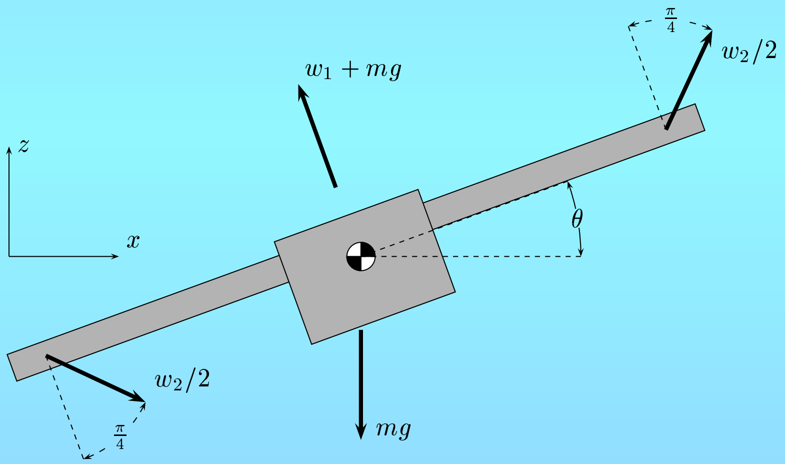
$$w_i = v_i(t,q) + \frac{1}{\epsilon} u_i\left(\frac{t}{\epsilon}, t\right),$$

$$v_i(t,q) = z_i^d(t) + \frac{1}{2}\sum_j \alpha_{ji}(q)\left(j - 1 + \sum_{k=j+1}^{m} (z_{jk}^d(t))^2\right),$$

$$u_i(\tau,t) = -\sum_{k=1}^{i-1} \psi_{N(k,i)}(\tau) + \sum_{k=i+1}^{m} z_{ik}^d(t)\psi_{N(i,k)}(\tau).$$

2.3 The PVTOL model

Simple planar vertical takeoff and landing aircraft model.



Motion planning strategies 2. Oscillatory control

Equations of motion,

$$\begin{aligned}
\dot{x} &= \cos\theta v_x - \sin\theta v_z \\
\dot{z} &= \sin\theta v_x + \cos\theta v_z \\
\dot{\theta} &= \omega \\
\dot{v}_x &= (-k_1/m)v_x - g\sin\theta + v_z\omega + (1/m)w_2 \\
\dot{v}_z &= (-k_2/m)v_z - g(\cos\theta - 1) - v_x\omega + (1/m)w_1 \\
\dot{\omega} &= (-k_3/J)\omega + (h/J)w_2
\end{aligned}$$

Control w_1 ➡ body vertical force minus gravity

Control w_2 ➡ coupled forces on the wingtips with a net horizontal component

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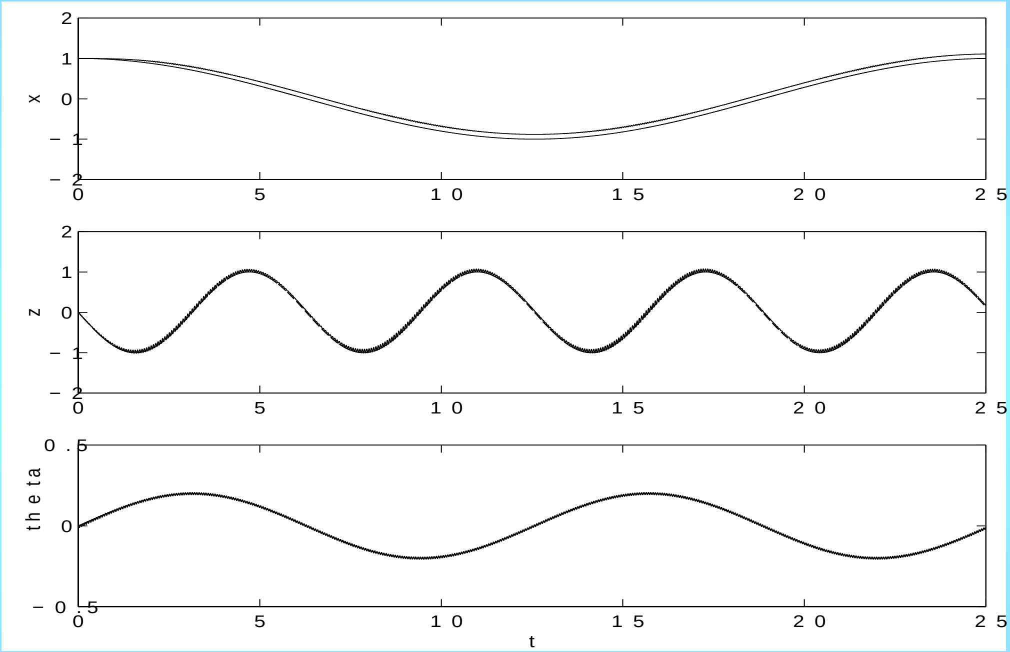


Figure 1: Tracking for the PVTOL model with $\epsilon = .01$.

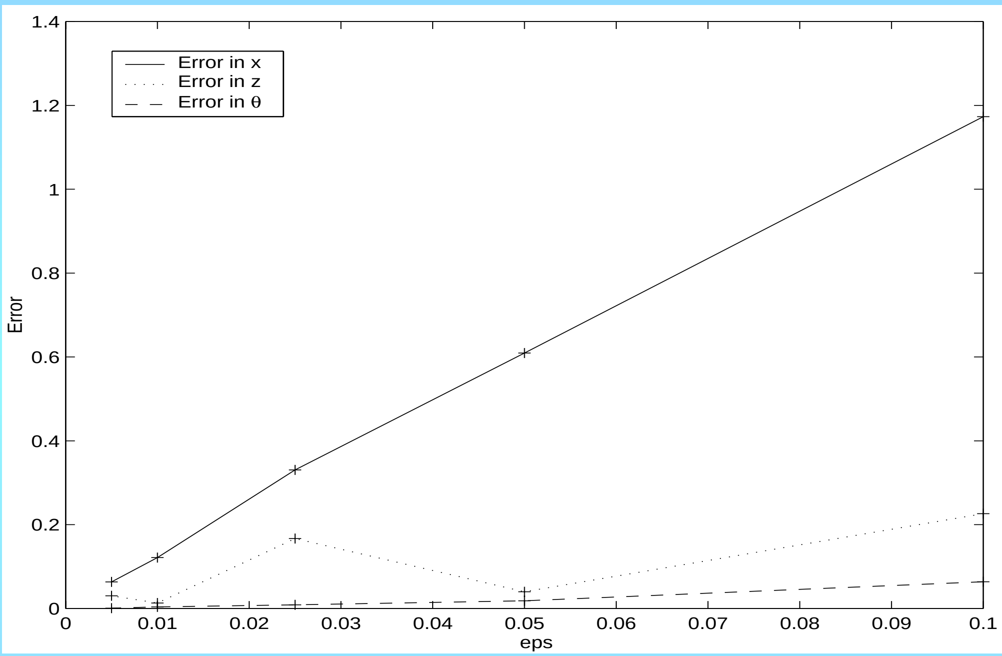


Figure 2: Tracking errors for the PVTOL model at $t = 10$.

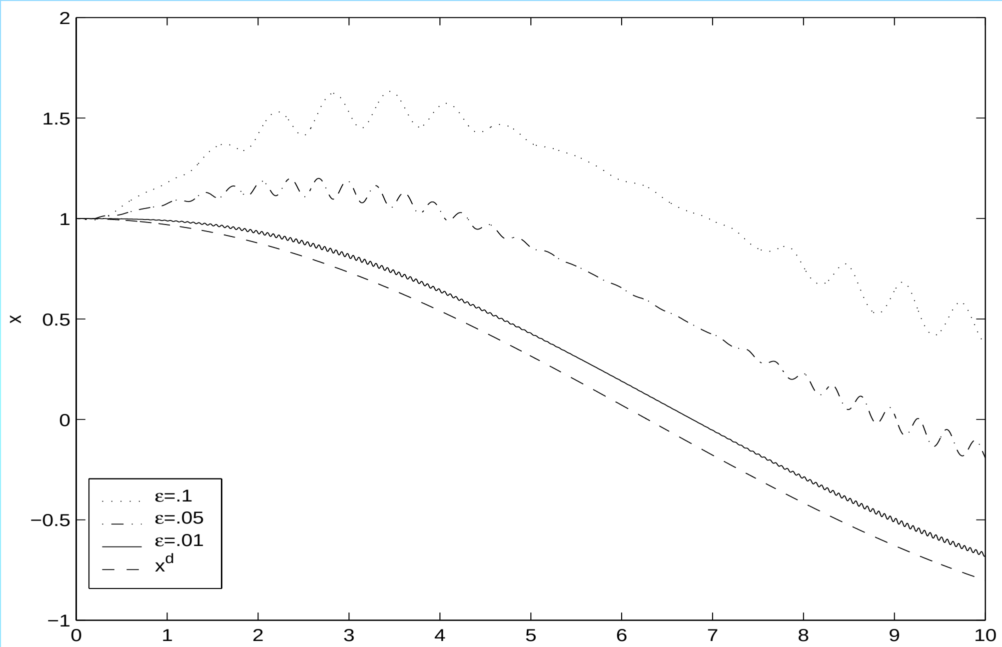


Figure 3: Tracking in the horizontal displacement of the PVTOL model.

3 Local planning via kinematic controllability

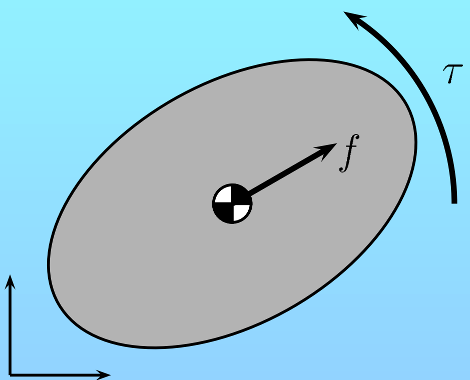
Piano-mover problem for underactuated mechanical system

(i) actuator failure

(ii) lighter design with less number of actuators

(iii) simplest case in which dynamics plays a role

motion planning instead of path planning



Simple example: body with one force through center of mass and one torque

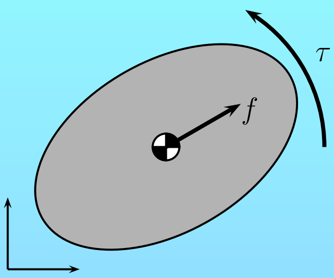
$$\ddot{\theta} = \tau$$
$$\ddot{x} = (\cos\theta) f$$
$$\ddot{y} = (\sin\theta) f$$

Three components

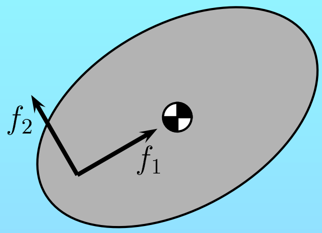
(i) Can follow any straight line and can turn (2 preferred velocity fields)

(ii) Controllable via these two motions

(iii) Planning via inverse kinematic



Ok



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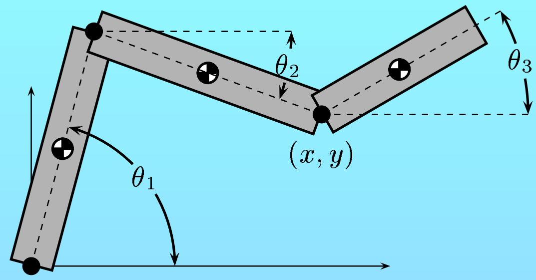


?

3.1 Kinematic systems

When a second order system can be realized as a kinematic one?

$$\nabla_{\dot q(t)}\dot q(t) = \sum_i u_i(t) Y_i(q)\,,\ q(0) = q_0 \quad \Rightarrow \quad \dot q(t) = \sum_i v_j(t) X_j(q)$$

Necessarily $\dot q(0) \in \mathrm{span}\{X_j\}$.

Theorem [Lewis, 1999]

The solutions $q(t)$ can be realized by the first-order system if and only if

- $\mathrm{span}\{X_j\} = \mathrm{span}\{Y_i\}$ and
- $\mathcal{I}$ is geodesically invariant: $\langle X : Y \rangle \in \mathcal{I}$, $\forall X, Y \in \mathcal{I}$.

From the motion planning point of view, great simplification!

Examples of "kinematic" systems: rolling disk, hopping robot.

However there are many which do not verify such strong conditions. So we re-state the question as

When can a second order system follow the solution of a first order?

Search for decoupling vector fields W describing 1st order ODEs whose time-scaled flow is solutions to (forced) 2nd order ODEs

Otherwise said

The curve $q(t) = \Phi^W_{0,s(t)}(q_0)$ can be "executed" by the underactuated mechanical control system for any time reparametrization $s(t)$.

Call $q(t)$ kinematic motion.

3.2 Decoupling vector fields and kinematic controllability

Bullo & Lynch, 2001

Theorem

The vector field W is decoupling $\iff$ $W \in \mathcal{I}$ and $\nabla_W W \in \mathcal{I}$.

System is kinematically controllable if it is controllable by means of kinematic motions.

Theorem

System is kinematically controllable if LARC on decoupling vector fields

Obs: kinematic controllabilty $\Rightarrow$ STLCC.

Idea for motion planning: concatenate motion along integral curves of decoupling vector fields to reach desired configuration.

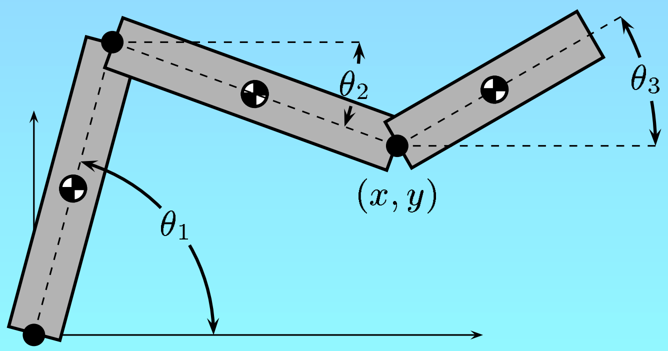
Let $q(t) = \Phi^W_{0,s(t)}(q_0)$, and W decoupling

$$\nabla_{\dot{q}(t)}\dot{q}(t) = \ddot{s}(t)W + \dot{s}^2\nabla_W W = \sum_i u_i(t)Y_i(q(t))$$

This way we choose the $u_i(t)$ so that $q(t)$ is realized by the affine connection control system.

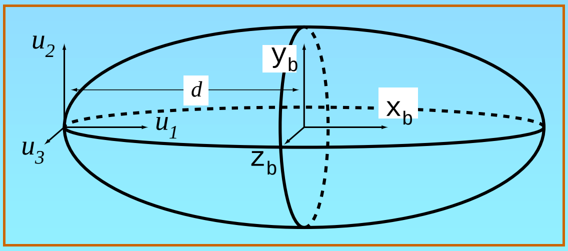
Note that vector fields in the input distribution which are not decoupling also play a role to "make" others decoupling.

Ex #1: Three link planar manipulator with passive link



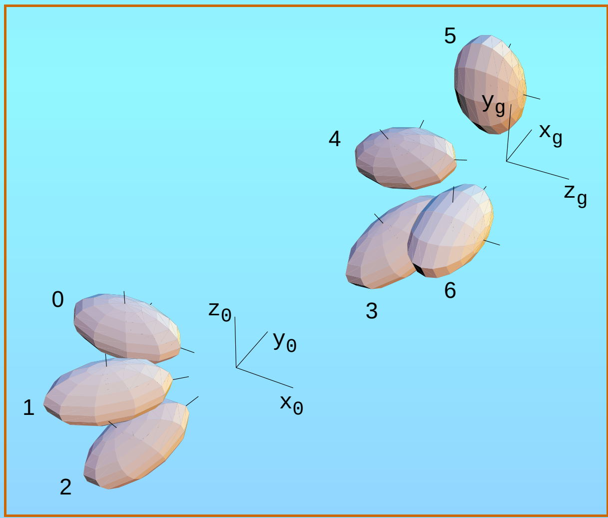
Actuator configuration	Decoupling vector fields	Kinematically controllable
(0,1,1)	2	yes
(1,0,1)	2	yes
(1,1,0)	2	yes

Ex #2: A three-dimensional aerospace vehicle with three forces

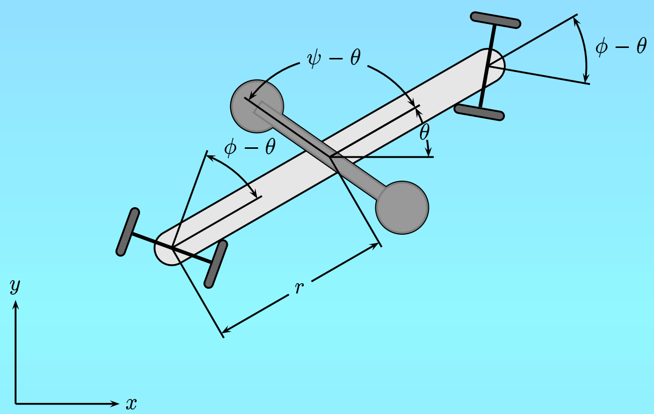


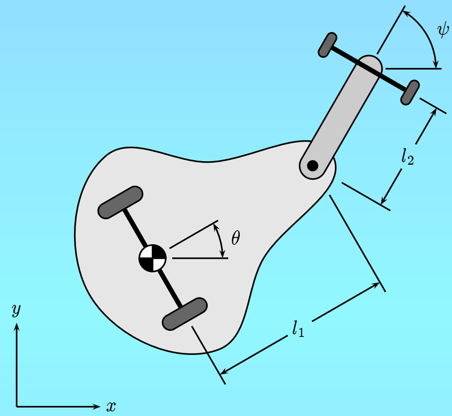
Kinematically controllable via
body-fixed constant velocity fields

Since invariant vector fields
decoupled trajectory planning
can be done
via inverse kinematic



Ex #3: The snakeboard and the roller racer





(i) snakeboard is kinematically controllable

(ii) roller racer is not:

(a) single input Y such that $\nabla_Y Y \notin \text{span}\{Y\}$

(b) moves forward using zero mean (cyclic) input

How to turn the racer into a kinematically controlable system?

$$\nabla_{\dot{q}}\dot{q} = Y(q)\left(u_1(t) + \frac{v\,(t/\epsilon)}{\epsilon\sqrt{\lambda}}u_2(t)\right)$$

$v(t)$ is T-periodic and cyclic

⬇

$$\nabla_{\dot{q}}\dot{q} = Y(q)u_1(t) + \langle Y : Y\rangle(q)\,u_2^2(t)$$

Hence, the roller racer would be kinematically controllable as $\epsilon \to 0$, provided $u_2^2(t)$ assumes positive and negative values

4 Conclusions

Differential geometric viewpoint on motion planning problem: affine connection formalism is well-suited for modeling, analysis and design.

Future work: further exploit geometric structure in development of motion planning strategies.

- Higher-order averaging analysis: application to stabilization and tracking.
- Kinematic controllability and related ideas for steering and path planning.
- Controllability tests for systems with unilateral inputs