

# COMMENSURATIONS AND SUBGROUPS OF FINITE INDEX OF THOMPSON'S GROUP $F$

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ABSTRACT. We determine the (abstract) commensurator  $\text{Com}(F)$  of Thompson's group  $F$  and describe it in terms of piecewise linear homeomorphisms of the real line and in terms of tree diagrams. This implies that  $\text{Com}(F)$  is not finitely generated and enables us to determine which subgroups of finite index in  $F$  are isomorphic to  $F$ . We also prove that the commensurator group is a subgroup of the quasi-isometry group of  $F$ .

## INTRODUCTION

Thompson's groups have been extensively studied since their introduction by Thompson in the 1960s. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see [2], for instance). The excellent introduction to Thompson's groups [4] is recommended to the interested reader, to find proofs of the background results we claim without proof.

Automorphisms for Thompson's group  $F$  were studied by Matthew Brin in [1], where a key theorem by McCleary and Rubin [8] is used to realize each automorphisms as conjugation by a piecewise linear map. A generalization is obtained when one considers commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under the equivalence relation of passing to a smaller subgroup), called the commensurator group.

We extend Brin's results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of  $\infty$  which we use to find the algebraic structure of  $\text{Com}(F)$ .

Commensurators have been studied before (see [7] and [5] for instance) due to their relation with quasi-isometries. Also, Thompson's group occurs naturally in the commensurators of branch groups [9]. The only quasi-isometries of  $F$  known to date were automorphisms. This paper provides a vast array of examples of quasi-isometries, since all commensurations are quasi-isometries, and it is proved in Section 5 that the commensurator group embeds into the quasi-isometry group.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of  $F$ . In Section 3 the main result about the commensurator is stated and proved, and in the next section its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5. The final sections of the paper are

devoted to infinite binary tree diagrams and how they can be used to understand the commensurations in a combinatorial way.

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## 1. DEFINITIONS

Let  $P$  denote the group of all homeomorphisms  $f$  from  $\mathbb{R}$  to itself that

- (1) are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of  $f'$ ),
- (2) use only slopes that are integral powers of 2,
- (3) have their breakpoints in the set  $\mathbb{Z}[\frac{1}{2}]$ , and
- (4) satisfy  $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$ .

It is easy to check that each element  $f$  of  $P$  actually satisfies  $f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$  and that  $P$  has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by  $P_+$ . The quotient  $P/P_+$  is generated by the image of the homeomorphism  $\tau : t \mapsto -t$ .

When  $\mathcal{P}$  is a property of maps, then we call an element  $f$  of  $P$  *eventually  $\mathcal{P}$*  if  $f$  satisfies  $\mathcal{P}$  for all  $t \in \mathbb{R}$  with  $|t| > M$  for some  $M > 0$ ; here  $|t|$  denotes the absolute value of  $t$ . For example,  $f \in P_+$  is eventually affine if there exist  $M, l, r \in \mathbb{R}$ ,  $M > 0$ , so that  $f(t) = t + r$  for all  $t > M$  and  $f(t) = t + l$  for all  $t < -M$ . Notice that  $l$  and  $r$  may well be different. We call  $f$  *integrally affine* if  $f(t) = \varepsilon t + p$  for some integer  $p$  and  $\varepsilon \in \{\pm 1\}$ .

It is well-known that Thompson's famous group  $F$ , which he introduced in [11], is isomorphic to the subgroup of  $P_+$  consisting of all eventually integrally affine elements, cf. [4]. It is now easy to see that the commutator subgroup  $F'$  of  $F$  consists of all eventually trivial elements of  $P_+$ , i.e.  $f(t) = t$ . This group is denoted by  $BPL_2(\mathbb{R})$  in [1], where  $B$  stands for bounded support.

We call  $f : \mathbb{R} \rightarrow \mathbb{R}$  *periodically affine* if  $f(t + p) = f(t) + q$  for some non-zero  $p, q \in \mathbb{R}$ , and  $f$  is *integrally periodically affine* if  $p$  and  $q$  are integers. Note that all integrally affine maps are integrally periodically affine with  $p = q$ .

## 2. FINITE-INDEX SUBGROUPS OF $F$

Let  $f$  be an element of  $F$ , which, we know is eventually integrally affine, so there exists two integers  $l, r$  and a real number  $M > 0$  such that  $f(t) = t + r$  for  $t > M$  and  $f(t) = t + l$  for  $t < -M$ . The two numbers  $l$  and  $r$  are precisely the components of the image of  $f$  in  $\mathbb{Z} \times \mathbb{Z}$  under the abelianization map. The subgroups of finite index of  $F$  are in one-to-one correspondence with those of its abelianized by the following result.

**Proposition 2.1.** *Let  $H$  be a subgroup of  $F$  of finite index. Then  $H$  contains  $F'$ , the commutator subgroup of  $F$ , and hence  $H$  is normal in  $F$ . Moreover,  $H' = F'$ .*

*Proof.* Since  $F$  is finitely generated,  $H$  has only finitely many conjugates in  $F$  and the intersection of all of them,  $K$  say, is normal and of finite index in  $F$ . Consider  $K \cap F'$ , which is thus normal and of finite index in  $F'$ . Hence, since  $F'$  is simple and infinite, we conclude that  $K \cap F' = F'$  and  $F' \subset K \subset H$ .

Hence  $H$  is normal in  $F$ . The final claim follows from the fact that  $H'$  is contained in  $F'$  but also characteristic in  $H$  and hence normal in  $F$ , whence  $F' \subset H'$ .  $\square$

From this fact we deduce that the finite-index subgroups of  $F$  are in bijection with those of  $\mathbb{Z} \times \mathbb{Z}$ . There is a distinguished family among these, namely, the subgroups  $p\mathbb{Z} \times q\mathbb{Z}$ . We denote by  $[p, q]$ ,  $p, q \in \mathbb{Z}$ , the preimage in  $F$  under the abelianization homomorphism of the subgroup  $p\mathbb{Z} \times q\mathbb{Z}$  of  $\mathbb{Z} \times \mathbb{Z}$ . Thus  $F = [1, 1]$  and  $F' = [0, 0]$ . These subgroups will be of interest later.

### 3. THE COMMENSURATOR GROUP

As mentioned before, a *commensuration* of a group  $G$  is an isomorphism  $\alpha : A \rightarrow B$ , where  $A$  and  $B$  are subgroups of  $G$  of finite index. Two commensurations  $\alpha$  and  $\beta$  are equivalent if they agree on some subgroup of finite index in  $G$ , which allows compositions to take place, because two isomorphisms

$$\alpha : A \longrightarrow B \quad \beta : C \longrightarrow D$$

can be composed when restricted to a smaller subgroup, namely, the composition  $\beta \circ \alpha$  will be defined on  $\alpha^{-1}(B \cap C)$ . The set of all commensurations of  $G$  modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of  $G$*  which we denote by  $\text{Com}(G)$ . If  $G$  is a subgroup of the group  $H$ , then the (relative) commensurator of  $G$  in  $H$ ,  $\text{Com}_H(G)$ , consists of all elements  $h$  of  $H$  for which  $G \cap G^h$  has finite index in both  $G$  and  $G^h$ ; here  $G^h = h^{-1}Gh$ .

The main result of this paper is the following.

**Theorem 3.1.** *The commensurator of  $F$  is isomorphic to  $\text{Com}_P(F)$ , which consists of all eventually integrally periodically affine elements (of  $P$ ).*

*Proof.* The strategy of the proof is to find a large group where  $F$  is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group  $P$  plays this role in the case of  $F$ .

To understand this fact we need some definitions and one of the main results of [8]. Let  $(L, <)$  be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup  $G$  of  $\text{Aut}(L)$  is *locally moving* if for every interval  $I$  there exists a nontrivial element  $g \in G$  which acts as the identity on  $L \setminus I$ . Finally,  $G$  is  *$n$ -interval-transitive* if for every pair of sequences of intervals  $I_1 < \dots < I_n$  and  $J_1 < \dots < J_n$  there exists  $g \in G$  such that  $I_k^g \cap J_k \neq \emptyset$  for  $1 \leq k \leq n$ . Below,  $\overline{L}$  denotes the Dedekind completion of  $L$  which is assumed to have no endpoints.

**Theorem 3.2** (Rubin-McCleary, [8]). *Assume  $(L_i, <)$  is a dense linear order without endpoints and let  $G_i \subset \text{Aut}(L_i)$  be locally moving and 2-interval transitive,  $i = 1, 2$ . Suppose that  $\alpha : G_1 \rightarrow G_2$  is an isomorphism. Then there is a monotonic bijection  $\tau : \overline{L}_1 \rightarrow \overline{L}_2$  which induces  $\alpha$ , that is,  $g^\alpha = \tau^{-1}g\tau$  for every  $g \in G_1$ ; and  $\tau$  is unique.*

Observe that being locally moving and 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

Now view  $\mathbb{Z}[\frac{1}{2}]$  as a dense linear order and  $F$  as the eventually integrally affine elements of  $P_+$ . Let  $\alpha : A \rightarrow B$  be a commensuration of  $F$ . By the lemma, both  $A$  and  $B$

contain  $F'$  which is (obviously) locally moving and 2-interval transitive [1, Lemma 2.1]. So Theorem 3.2 tells us that  $\alpha$  is induced by conjugation with a unique element of  $\text{Homeo}(\mathbf{R})$ . This yields an injective homomorphism  $\Psi : \text{Com}(F) \rightarrow \text{Homeo}(\mathbf{R})$ .

Next, we show that the image of  $\Psi$  is in fact contained in  $P$ . By Proposition 2.1 each commensuration of  $F$  induces an automorphism of  $F'$ . In other words, the image of  $\Psi$  is contained in  $N_{\text{Homeo}(\mathbf{R})}(F')$ , the normalizer of  $F'$  in  $\text{Homeo}(\mathbf{R})$ . But this normalizer is equal to  $P$  by [1, Theorem 1]. The existence and uniqueness statements in Theorem 3.2 now imply that  $\Psi$  is an isomorphism between  $\text{Com}(F)$  and  $\text{Com}_P(F)$ , which proves the first part of Theorem 3.1.

Let  $\alpha \in \text{Com}(F)$  and choose positive integers  $p$  and  $q$  so large that  $\alpha$  is defined on the subgroup  $[p, q]$ , that is  $[p, q]^\alpha$ , the image of  $[p, q]$  under  $\alpha$ , is contained in  $F$ . By what was said above, we can view  $\alpha$  as (conjugation by) an element of  $P$ . So for  $f \in [p, q]$  we must find  $f^\alpha = \alpha^{-1}f\alpha$  to be eventually integrally affine. Suppose for a moment that  $\alpha$  is order preserving and that  $f(t) = t + kq$  for  $t \gg 0$ , where  $k \in \mathbf{Z}$ . Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some  $r \in \mathbf{Z}$ . In other words,  $\alpha^{-1}(t+r) = \alpha^{-1}(t) + s$  for some integers  $r$  and  $s$  and all  $t \gg 0$ . Since  $f$  was arbitrary, we may assume that  $k \neq 0$ , which implies that  $s \neq 0$ , and hence also  $r \neq 0$ . Therefore  $\alpha^{-1}$ , and hence  $\alpha$ , must be integrally periodically affine near infinity. A similar calculation holds for  $t \ll 0$  and also when  $\alpha$  is order reversing. Consequently, each commensuration of  $F$  must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine  $\beta \in P$  induces a commensuration of  $F$  by conjugation. Suppose  $\beta(t+p) = \beta(t) + q$  for  $t \gg 0$  and  $\beta(t+p') = \beta(t) + q'$  for  $t \ll 0$ ,  $p, q, p', q' \in \mathbf{Z} \setminus \{0\}$ . Let  $U = [p', p]$  if  $\beta$  is order preserving and set  $U = [p, p']$  otherwise. Then for  $f \in U$ , we have

$$f^\beta(t) = \begin{cases} \beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\ \beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0 \end{cases}$$

where  $k, k' \in \mathbf{Z}$  depend on  $f$ . Together with a similar argument for  $\beta^{-1}$  one easily sees that  $U^\beta = [q', q]$  or  $[q, q']$ , depending on whether  $\beta$  is order preserving or not. Theorem 3.1 is thus established.  $\square$

Some corollaries can be readily obtained from this result.

**Corollary 3.3.** *A subgroup  $U$  of  $F$  of finite index is isomorphic to  $F$  if and only if  $U = [p, q]$  for some positive integers  $p$  and  $q$ .*

*Proof.* Suppose  $U$  is a subgroup of finite index in  $F$ . If  $U$  is isomorphic to  $F$ , then there exists an eventually integrally periodically affine  $\alpha \in P$  with  $F^\alpha = U$  and calculations as above show that  $U$  must be of the form  $[p, q]$ . On the other hand, the final paragraph of the proof of the theorem read with  $p = p' = 1$  shows that  $[q', q]$  is isomorphic to  $F$  for every choice of positive integers  $q$  and  $q'$ . This completes the proof.  $\square$

Finally, since each subgroup of finite index in  $F$  contains  $[p, q]$  for some positive integers  $p$  and  $q$  (Proposition 2.1), we have the following results.

**Corollary 3.4.** *Every finite-index subgroup of  $F$  is virtually  $F$ .*

**Corollary 3.5.** *A group is commensurable with  $F$  if and only if it is a finite extension of  $F$ .*

#### 4. THE STRUCTURE OF $\text{Com}(F)$

Once we know the characteristics of the elements in  $\text{Com}(F)$  we can study its structure as a group. An element  $\alpha$  in  $\text{Com}(F)$  is eventually integrally periodically affine, so there exist positive integers  $p, p', q, q'$  and a real number  $M$  such that

$$\begin{aligned}\alpha(t + p) &= \alpha(t) + q, \text{ for } t > M \\ \alpha(t + p') &= \alpha(t) + q', \text{ for } t < -M.\end{aligned}$$

We need a lemma about affine functions, whose proof is elementary and left to the reader.

**Lemma 4.1.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be an integrally periodically affine map, and assume that there are integers  $i, i', j, j'$  such that for all  $t \in \mathbf{R}$  we have*

$$f(t + i) = f(t) + j \quad \text{and} \quad f(t + i') = f(t) + j'.$$

Then we have

$$f(t + r) = f(t) + s,$$

where

$$r = \gcd(i, i') \quad \text{and} \quad s = \gcd(j, j').$$

Furthermore, we have

$$\frac{i}{j} = \frac{i'}{j'}.$$

From this lemma, we see that the integers  $p, p', q, q'$  that are characteristic of an element of  $\text{Com}(F)$  are important and depend only on the element.

Recall that  $\text{Com}(F)$  has a subgroup of index 2, denoted  $\text{Com}^+(F)$ , formed by the commensurations induced by conjugation of a piecewise-linear map which preserves the orientation of  $\mathbf{R}$ .

**Proposition 4.2.** *There exists a homomorphism*

$$\Phi : \text{Com}^+(F) \longrightarrow \mathbf{Q}^* \times \mathbf{Q}^*$$

defined by

$$\Phi(f) = \left( \frac{p}{q}, \frac{p'}{q'} \right).$$

Here  $\mathbf{Q}^*$  denotes the multiplicative group of the non-zero rational numbers.

The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map are what happens at both ends, eventually near  $-\infty$  and eventually near  $+\infty$ . The two numbers  $p/q$  and  $p'/q'$  measure the “rate of growth” of the map at both ends.

A corollary of this result is that, as expected,  $\text{Com}(F)$  is infinitely generated.

Let  $H$  be the kernel of this map. An element of  $H$  will be a map  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that there exists two integers  $p, p'$  and a real number  $M$  such that

$$\begin{aligned}\alpha(t+p) &= \alpha(t) + p, \text{ for } t > M \\ \alpha(t+p') &= \alpha(t) + p', \text{ for } t < -M.\end{aligned}$$

And observe that  $p$  does not have to be necessarily equal to 1. A map which satisfies  $f(t+p) = f(t) + p$  can be seen as an element of the group  $T_{2,p}$ , see [6]. This group is denoted by  $T(p, \mathbb{Z}[1/2], 2\mathbb{Z})$  in [10].

The group  $T_{2,p}$  is the group of the piecewise linear maps of the circle of circumference  $p$ , considered as the interval  $[0, p]$  with identified endpoints. These piecewise linear maps must have all breakpoints in dyadic integers inside  $[0, p]$ , and all the slopes of the linear parts are powers of 2. (This is what the index 2 means in Higman's notation.)

Observe that if  $p|q$ , then a circle of length  $q$  can be wrapped  $q/p$  times over a circle of length  $p$ . So, if a map satisfies  $f(t+p) = f(t) + p$ , then it also satisfies  $f(t+q) = f(t) + q$ . Hence these groups admit a family of embeddings

$$i_{pq} : T_{2,p} \longrightarrow T_{2,q}$$

for any pair of positive integers  $p$  and  $q$  such that  $p|q$ . Clearly the family  $\{T_{2,p}, i_{pq}\}$  is a directed family, and we define

$$\mathbb{T} = \varinjlim_p T_{2,p}.$$

Elements of  $\mathbb{T}$  are piecewise linear maps with dyadic breaks and slopes powers of 2 that are defined on a circle of some circumference  $p$ . Or alternately, elements of  $P$  such that  $f(t+p) = f(t) + p$  for some integer  $p$ . The group  $\mathbb{T}$  is an infinitely generated subgroup of  $P$ .

From this we infer that an element of  $H$  is eventually at both ends an element of some  $T_{2,p}$ , and hence of  $\mathbb{T}$ . This yields a homomorphism

$$\Psi : H \longrightarrow \mathbb{T} \times \mathbb{T}.$$

Finally, observe that if an element  $h$  of  $H$  is mapped to the identity by  $\Psi$ , then its eventually integrally periodically affine part at both ends is the identity in  $T_{2,p}$ , hence it is a map with slope 1, of the type  $t \mapsto t + r$ . But to be the identity in  $T_{2,p}$ ,  $r$  has to be an integer, and furthermore, a multiple of  $p$ , because if not, it induces a nontrivial rotation on the circle of length  $p$ . Hence it is an element of  $F$ . This proves the final step in the structure theorem from  $\text{Com}(F)$ .

**Theorem 4.1.** *The group-theoretic structure of the commensurator group of Thompson's group  $F$  is given by the following three short exact sequences:*

$$\begin{aligned}1 &\longrightarrow \text{Com}^+(F) \longrightarrow \text{Com}(F) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1 \\ 1 &\longrightarrow H \longrightarrow \text{Com}^+(F) \xrightarrow{\Phi} \mathbb{Q}^* \times \mathbb{Q}^* \longrightarrow 1 \\ 1 &\longrightarrow F \longrightarrow H \xrightarrow{\Psi} \mathbb{T} \times \mathbb{T} \longrightarrow 1\end{aligned}$$

## 5. COMMENSURATORS AS QUASI-ISOMETRIES

Let  $G$  be a finitely generated group. The group of quasi-isometries  $QI(G)$  has as elements the quasi-isometries modulo bounded distance, i.e., two quasi-isometries  $f$  and  $g$  are considered equivalent if there exists a number  $M > 0$  such that  $d(f(t), g(t)) \leq M$  for all  $t$ .

It is well known that the commensurator group admits a map in the quasi-isometry group, since all commensurations can be trivially extended to quasi-isometries. The result we want to prove in this section is that for Thompson's group  $F$ , this map is one-to-one.

**Theorem 5.1.** *The natural homomorphism  $\text{Com}(F) \longrightarrow QI(F)$  is injective.*

*Proof.* We begin with an elementary lemma.

**Lemma 5.2.** *Given an element  $\tau \in P$  which is different from the identity, there exist two intervals  $I$  and  $J$  of the real line, whose endpoints are dyadic integers, with  $\tau(I) = J$ , and such that  $I \cap J = \emptyset$ .*

The case when the slope of  $\tau$  is always 1 or -1 is trivial. A map  $t \mapsto t + k$  has a small interval (of length less than  $k$ ) whose image is disjoint from it. If  $\tau = -Id$  the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope  $\pm 2^i$  with  $i \neq 0$ . Assume without loss of generality (maybe taking  $\tau^{-1}$  or by composing with  $-Id$ ) that  $i > 0$  and the slope is positive. Hence there are two intervals  $[a, b]$  and  $[c, d]$  such that  $\tau(a) = c$  and  $\tau(b) = d$  and also  $d - c = 2^i(b - a)$ . It is possible that  $[a, b]$  and  $[c, d]$  overlap, but since  $[c, d]$  is much larger than  $[a, b]$  (at least twice the size), we can choose as  $J$  a small interval inside  $[c, d]$  which is disjoint from  $[a, b]$ . By construction, the preimage  $I$  of  $J$  is in  $[a, b]$ , and hence  $I$  and  $J$  are disjoint. This finishes the proof of the lemma.

Take now  $\tau \in \text{Com}(F)$  which is not the identity. By the lemma above, it has intervals  $I$  and  $J$  satisfying the conditions stated above. Consider now all elements of  $F$  whose support (that is, the part where they are not the identity) is contained in  $I$ . Those elements form a subgroup which is isomorphic to  $F$  itself. Let  $f$  be one such element. Since its support is inside  $I$ , its image under the commensuration  $\tau$ , that is,  $f^\tau = \tau \circ f \circ \tau^{-1}$ , has support inside  $J$ .

Hence, the distance (inside  $F$ ) from  $f$  to  $f^\tau$  is given by the distance from the identity to the element  $f^\tau f^{-1}$ . But this element has its support inside  $I \dot{\cup} J$ , and the two parts are independent from each other (one given by  $f$  and the other one by  $f^\tau$ ). By [3], this subgroup (elements with support in  $I \dot{\cup} J$ ) is quasi-isometrically embedded in  $F$ . Hence, we can take elements  $f_n$  with support inside  $I$  with arbitrarily large norm, and hence  $f_n^\tau f_n^{-1}$  is also arbitrarily long. This proves that  $\tau$ , when considered as a quasi-isometry, is not at bounded distance from the identity.  $\square$

## 6. INFINITE TREE DIAGRAMS

A particularly useful way to describe Thompson's group  $F$  has been widely used, namely, pairs of binary trees. A (finite) binary tree represents a subdivision of the interval  $[0, 1]$

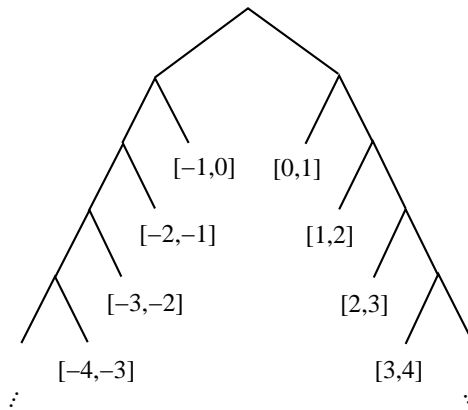


FIGURE 1. The backbone for trees with bounded interior depth.

into subintervals, whose lengths are powers of 2. An element of the group is then represented by a pair of trees, one for the source and one for the target of the map.

Generalizing this construction, we will give a binary tree interpretation of  $\text{Com}^+(F)$  in terms of binary trees, infinite in this case, but periodic, in some sense that will be explained below.

We will consider binary trees whose left and right sides are infinitely long. If we consider both sides of the tree a “backbone” from where all the other carets hang, each of the leaves will represent an interval of length one with integer endpoints; see Fig. 1. Each backbone leaf represents an interval of the type  $[k, k + 1]$ ,  $k$  an integer.

We want to represent finite subdivisions of the integer intervals, so we need to allow a finite subtree hanging from a leaf in the backbone. This leads to the following definition.

**Definition 6.1.** *A tree with bounded interior depth is a binary tree, possibly infinite, whose only possible infinite branches are the leftmost and rightmost branches. Each interior leaf is at bounded depth with respect to the root.*

The root of the tree marks the point where zero is, and each of the subtrees on a backbone leaf represents the subdivision of the corresponding interval  $[k, k + 1]$ .

As usual a *caret* is a nonleaf vertex together with its two descending edges.

Using trees with bounded interior depth we can represent any subdivision of the real line into intervals with length an integral power of 2, and whose endpoints are dyadic integers. We would like to use trees with bounded interior depth to represent elements of  $P_+$ . The two subdivisions of the source and target can be represented by trees with bounded interior depth, but we need a marking to indicate how intervals map to each other. A *marked tree with bounded interior depth* is a tree with bounded interior depth where one leaf has been selected (or marked). Armed with these trees we have the following result.

**Proposition 6.2.** *Every element of  $P_+$  can be represented by a pair of marked trees with bounded interior depth. Such pairs will be called bounded interior tree pair diagrams.*

The proof is clear: the trees represent the subdivisions of the real line, such that each interval is mapped linearly to another one, and the two markings indicate two paired

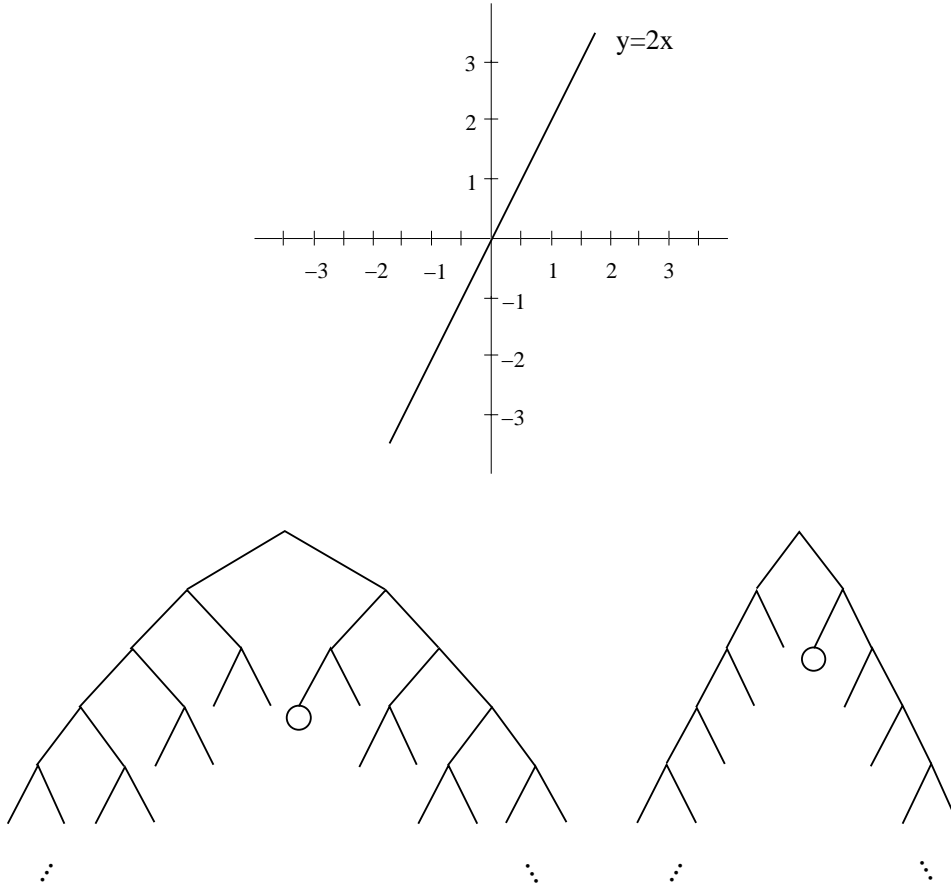


FIGURE 2. A diagram for an element in  $P$ . The two circles represent the marking, indicating that those two intervals map to each other.

intervals. Continuity forces the other intervals to pair in order-preserving fashion. See Fig. 2 for an example.

Two bounded interior tree pair diagrams can represent the same element of  $P_+$ , if two carets are mapped to each other in such a way that the subdivisions they represent can be eliminated, or if the markings appear in different but equivalent places. Thus an element of  $P_+$  is represented by an equivalence class of bounded interior tree pair diagrams, one of which is minimal if no carets can be eliminated.

Elements of  $F$  are precisely those where all but finitely many of the backbone leaves have empty subtrees, and hence can be eliminated. The resulting diagram is the usual (finite) binary tree representation for  $F$ . There is no need of a marking in that case, since only a finite number of leaves appear and they map preserving the order.

The reader who is familiar with this binary tree representation for Thompson's groups will have no problem extending it to this setting of piecewise linear maps of the real line. We introduce it here because elements of  $\text{Com}(F)$  will have a particularly easy description in this setting.

## 7. PERIODIC TREE DIAGRAMS

The elements of  $P_+$  which give commensurations are those which are eventually integrally periodically affine. The condition  $f(t+p) = f(t) + q$  which has to be satisfied for  $t$  larger than some  $M$ , implies that, for one of such  $t > M$ , all intervals  $[t+pk, t+p(k+1)]$ , for all integers  $k \geq 0$ , map the same way into an interval of length  $q$ , and hence, they all need to be subdivided in the exact same way.

This leads to the concept of *eventually periodic* trees. Periodic trees are trees of bounded interior depth, but such that subtrees repeat themselves.

**Definition 7.1.** *A  $p$ -periodic tree is a tree with bounded interior depth, such that for all integers  $k$ , the subtree hanging from the backbone leaf  $[k, k+1]$  is the same as the one in  $[k+p, k+p+1]$ .*

A  $p$  periodic tree is said to have a period of  $n$  leaves, if the trees on the backbone leaves  $[k, k+1]$ ,  $[k+1, k+2]$ ,  $\dots$ ,  $[k+p-1, k+p]$  have a total of  $n$  leaves among them.

The first tree in Fig. 3, for instance, is 3-periodic near  $\infty$  with a period of 6 leaves. The second tree, however, is 1-periodic with 2 leaves, needing three of these 1-periods to match the 3-period of the left tree.

Since elements of the commensurator are those which are periodic only eventually, we need a final definition.

**Definition 7.2.** *A marked pair of bounded interior depth is called eventually periodic if there exist positive integers  $M, p, p', q, q', n, n'$  such that:*

- (1) *The source tree is  $p$ -periodic with a period of  $n$  leaves for all backbone leaves  $[k, k+1]$  with  $k > M$ ; and  $p'$  periodic with a period of  $n'$  leaves for  $k < -M$ .*
- (2) *The source tree is  $q$ -periodic with a period of  $n$  leaves for all backbone leaves  $[k, k+1]$  with  $k > M$ ; and  $q'$  periodic with a period of  $n'$  leaves for  $k < -M$ .*

Observe that the number of leaves in the periods are the same for both trees. The idea is that in the neighborhood of  $\infty$ , the condition  $f(t+p) = f(t) + q$  translates into periodic trees, with periods  $p$  for the source and  $q$  for the target. Since intervals of length  $p$  have to be mapped to intervals of length  $q$ , the number of leaves for them has to be the same. And this phenomenon also appears (with different numbers) in a neighborhood of  $-\infty$ .

The result, whose proof is elementary in view of all these definitions, is that elements of the commensurator are defined by this kind of diagrams.

**Proposition 7.3.** *Every element in  $\text{Com}^+(F)$  admits a representative as an eventually periodic marked pair of trees with bounded interior depth.*

Composition of pairs of diagrams is obtained in the usual way by finding common subdivisions of the middle trees. It is a nice exercise on these diagrams to check that conjugation of an element of  $F$ , which belongs to the subgroup  $[p, p']$ , by an eventually periodic pair with  $p, p', q, q'$  as above, gives an element of the subgroup  $[q, q']$  of  $F$ .

Finally, remark that if one wishes to construct trees for the elements of  $\text{Com}(F)$ , i.e., including those reversing orientations, one only needs to introduce a second marking which indicates in which direction the leaves are mapped to the other. Everything else about periodic trees can be extended to this case in a straightforward way.

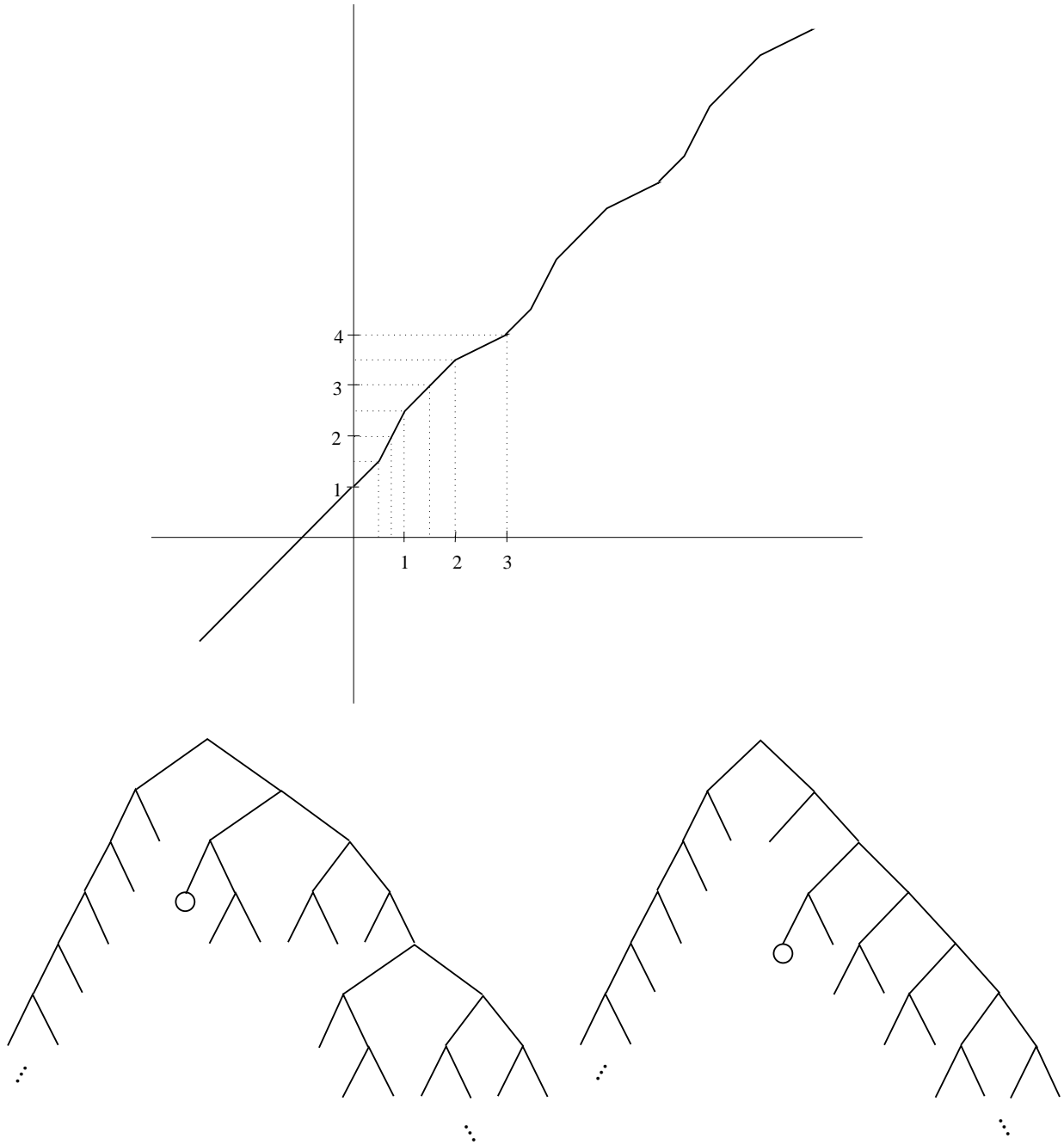


FIGURE 3. Another element in  $P_+$ .

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