

## Commensurations and Subgroups of Finite Index of Thompson’s Group $F$

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We determine the abstract commensurator  $\text{Com}(F)$  of Thompson’s group  $F$  and describe it in terms of piecewise linear homeomorphisms of the real line. We show  $\text{Com}(F)$  is not finitely generated and determine which subgroups of finite index in  $F$  are isomorphic to  $F$ . We also show that the natural map from the commensurator group to the quasi-isometry group of  $F$  is injective.

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### Introduction

Thompson’s groups have been extensively studied since their introduction by Thompson in the 1960s, despite the fact that Thompson’s account [7] appeared only in 1980. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see for example, Brown and Geoghegan [3]). Cannon, Floyd and Parry [4] give an excellent introduction to Thompson’s groups where many of the basic results used below are proven carefully.

Automorphisms for Thompson’s group  $F$  were studied by Brin [2], where a key theorem by McCleary and Rubin [6] is used to realize each automorphisms as conjugation by a piecewise linear map. Here, we generalize from automorphisms to commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under a natural equivalence relation involving passing to smaller yet still finite-index subgroups), called the commensurator group.

We classify finite-index subgroups of  $F$ , and then we extend Brin’s results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of  $\infty$  which we use to find the algebraic structure of the commensurator of  $F$ .

Commensurators have proven to be an effective tool for investigating quasi-isometries of a group to itself, and for effectively analyzing rigidity, particularly of lattices. In the case of  $F$ , the only quasi-isometries of  $F$  known previously were automorphisms. This paper provides a wide array of examples of quasi-isometries, since all commensurations are quasi-isometries, and we prove in Section 5 that the commensurator group embeds into the quasi-isometry group in the case of  $F$ .

Our approach is algebraic, but we note that elements of the commensurator of  $F$  can be represented by marked, infinite, eventually periodic, binary tree pair diagrams. We also note that recently Bleak and Wassink [1] have independently described the finite-index subgroups of  $F$ , using different methods.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of  $F$ . In Section 3 the main result about the commensurator is stated and proved, and in Section 4 its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5.

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## 1 Definitions

Let  $P$  denote the group of all homeomorphisms  $f$  from  $\mathbb{R}$  to itself that

- (1) are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of the derivative of  $f$ ),
- (2) use only slopes that are integral powers of 2,
- (3) have their breakpoints in the set  $\mathbb{Z}[\frac{1}{2}]$  and
- (4) satisfy  $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$ .

It is easy to check that each element  $f$  of  $P$  actually satisfies  $f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$  and that  $P$  has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by  $P_+$ . The quotient  $P/P_+$  is generated by the image of the homeomorphism  $\tau: t \mapsto -t$ .

Let  $f \in P$ . We call  $f$  *integrally affine* if  $f(t) = \varepsilon t + p$  for some integer  $p$  and  $\varepsilon \in \{\pm 1\}$ . We say  $f$  is *periodically affine* if  $f(t + p) = f(t) + q$  for some non-zero  $p, q \in \mathbb{R}$  and *integrally periodically affine* if  $p$  and  $q$  are integers. Note that all integrally affine maps are integrally periodically affine with  $q = \pm p$  depending on whether  $f$  is in  $P_+$  or not.

When  $\mathcal{P}$  is any of the above properties, then we call  $f$  *eventually  $\mathcal{P}$*  if  $f$  satisfies  $\mathcal{P}$  for all  $t \in \mathbb{R}$  with  $|t| > M$  for some  $M > 0$ ; here  $|t|$  denotes the absolute value of  $t$ . For example,  $f \in P_+$  is eventually integrally affine if there exist  $l, r \in \mathbb{Z}$ ,  $M \in \mathbb{R}$ ,  $M > 0$ , so that  $f(t) = t + r$  for all  $t > M$  and  $f(t) = t + l$  for all  $t < -M$ . Notice that  $l$  and  $r$  may well be different.

It is well-known that Thompson's group  $F$  is isomorphic to the subgroup of  $P_+$  consisting of all eventually integrally affine elements (see [4]). It is easy to see that the commutator subgroup  $F'$  of  $F$  consists of all eventually trivial elements of  $P_+$  (those where eventually  $f(t) = t$ ). This group is denoted by  $BPL_2(\mathbb{R})$  by Brin [2], where  $B$  stands for bounded support.

## 2 Finite-index Subgroups of $F$

Let  $f$  be an element of  $F$ . Since  $f$  is eventually integrally affine, there are two integers  $l, r$  and a real number  $M > 0$  such that  $f(t) = t + r$  for  $t > M$  and  $f(t) = t + l$  for  $t < -M$ . The two numbers  $l$  and  $r$  are precisely the two components of the image of  $f$  in  $\mathbb{Z} \times \mathbb{Z}$  under the abelianization map. The subgroups of finite index of  $F$  are in one-to-one correspondence with those of its abelianization  $\mathbb{Z} \times \mathbb{Z}$  by the following result.

**Proposition 2.1** *Let  $H$  be a subgroup of  $F$  of finite index. Then  $H$  contains  $F'$ , the commutator subgroup of  $F$ , and hence  $H$  is normal in  $F$ . Moreover,  $H' = F'$ .*

**Proof** Since  $F$  is finitely generated,  $H$  has only finitely many conjugates in  $F$  and the intersection of all of them,  $K$  say, is normal and of finite index in  $F$ . We consider  $K \cap F'$ , which is thus normal and of finite index in  $F'$ . Hence, since  $F'$  is simple and infinite, we conclude that  $K \cap F' = F'$  and  $F' \subset K \subset H$ .

Hence  $H$  is normal in  $F$ . The final claim follows from the fact that  $H'$  is contained in  $F'$  but also characteristic in  $H$  and hence normal in  $F$ , whence  $F' \subset H'$ .  $\square$

From this fact we deduce that the finite-index subgroups of  $F$  are in bijection with those of  $\mathbb{Z} \times \mathbb{Z}$ . There is a distinguished family among these—the subgroups  $p\mathbb{Z} \times q\mathbb{Z}$ . We denote by  $[p, q]$ ,  $p, q \in \mathbb{Z}$ , the preimage in  $F$  under the abelianization homomorphism of the subgroup  $p\mathbb{Z} \times q\mathbb{Z}$  of  $\mathbb{Z} \times \mathbb{Z}$ . Thus  $F = [1, 1]$  and  $F' = [0, 0]$ .

### 3 The Commensurator Group

As mentioned before, a *commensuration* of a group  $G$  is an isomorphism  $\alpha: A \rightarrow B$ , where  $A$  and  $B$  are subgroups of  $G$  of finite index. Two commensurations  $\alpha$  and  $\beta$  are equivalent if they agree on some subgroup of finite index in  $G$ . In view of this, the product  $\beta \circ \alpha$  of two commensurations

$$\alpha: A \rightarrow B \quad \text{and} \quad \beta: C \rightarrow D$$

is defined on  $\alpha^{-1}(B \cap C)$ . The set of all commensurations of  $G$  modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of  $G$*  which we denote by  $\text{Com}(G)$ . If  $G$  is a subgroup of the group  $H$ , then the (relative) commensurator of  $G$  in  $H$ ,  $\text{Com}_H(G)$ , consists of all elements  $h$  of  $H$  for which  $G \cap G^h$  has finite index in both  $G$  and  $G^h$ ; here  $G^h = h^{-1}Gh$ .

The main result of this paper is the following.

**Theorem 3.1** *The commensurator of  $F$  is isomorphic to  $\text{Comp}(F)$ , which consists of all eventually integrally periodically affine elements (of  $P$ ).*

The strategy of the proof is to find a large group where  $F$  is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group  $P$  plays this role in the case of  $F$ .

In order to explain this strategy, we need some definitions and one of the main results of McCleary and Rubin [6]. Let  $(L, <)$  be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup  $G$  of  $\text{Aut}(L)$  is *locally moving* if for every interval  $I$  there exists a nontrivial element  $g \in G$  which acts as the identity on  $L \setminus I$ . Finally,  $G$  is  *$n$ -interval-transitive* if for every pair of sequences of intervals  $I_1 < \dots < I_n$  and  $J_1 < \dots < J_n$  there exists  $g \in G$  such that  $I_k^g \cap J_k \neq \emptyset$  for  $1 \leq k \leq n$ . Below,  $\bar{L}$  denotes the Dedekind completion of  $L$  which is assumed to have no endpoints.

**Theorem 3.2** (McCleary–Rubin [6]) *Assume  $(L_i, <)$  is a dense linear order without endpoints and let  $G_i \subset \text{Aut}(L_i)$  be locally moving and 2-interval transitive,  $i = 1, 2$ . Suppose that  $\alpha: G_1 \rightarrow G_2$  is an isomorphism. Then there is a monotonic bijection  $\tau: \bar{L}_1 \rightarrow \bar{L}_2$  which induces  $\alpha$ , that is,  $g^\alpha = \tau^{-1}g\tau$  for every  $g \in G_1$ ; and  $\tau$  is unique.*

Being locally moving and having 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

**Proof of Theorem 3.1** View  $\mathbb{Z}[\frac{1}{2}]$  as a dense linear order and  $F$  as the eventually integrally affine elements of  $P_+$ . Let  $\alpha: A \rightarrow B$  be a commensuration of  $F$ . By Proposition 2.1, both  $A$  and  $B$  contain  $F'$  which is (obviously) locally moving and 2-interval transitive (see [2, Lemma 2.1]). So Theorem 3.2 tells us that  $\alpha$  is induced by conjugation with a unique element of  $\text{Homeo}(\mathbb{R})$ . This yields an injective homomorphism  $\Psi: \text{Com}(F) \rightarrow \text{Homeo}(\mathbb{R})$ .

Next, we show that the image of  $\Psi$  is in fact contained in  $P$ . By Proposition 2.1, each commensuration of  $F$  induces an automorphism of  $F'$ . In other words, the image of  $\Psi$  is contained in  $N_{\text{Homeo}(\mathbb{R})}(F')$ , the normalizer of  $F'$  in  $\text{Homeo}(\mathbb{R})$ . But this normalizer is equal to  $P$  by Theorem 1 of Brin [2]. The existence and uniqueness statements in Theorem 3.2 now imply that  $\Psi$  is an isomorphism between  $\text{Com}(F)$  and  $\text{Com}_P(F)$ , which proves the first part of Theorem 3.1.

Let  $\alpha \in \text{Com}(F)$  and choose positive integers  $p$  and  $q$  so large that  $\alpha$  is defined on the subgroup  $[p, q]$ , that is  $[p, q]^\alpha$ , the image of  $[p, q]$  under  $\alpha$ , is contained in  $F$ . By what was said above, we can view  $\alpha$  as conjugation by an element of  $P$ . So for  $f \in [p, q]$  we find  $f^\alpha = \alpha^{-1}f\alpha$  to be eventually integrally affine. Suppose for a moment that  $\alpha$  is order preserving and that  $f(t) = t + kq$  for  $t \gg 0$ , where  $k \in \mathbb{Z}$ . Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some  $r \in \mathbb{Z}$ . In other words,  $\alpha^{-1}(t+r) = \alpha^{-1}(t) + s$  for some integers  $r$  and  $s$  and all  $t \gg 0$ . Since  $f$  was arbitrary, we may assume that  $k \neq 0$ , which implies that  $s \neq 0$ , and hence also  $r \neq 0$ . Therefore  $\alpha^{-1}$ , and hence  $\alpha$ , must be integrally periodically affine near infinity. A similar calculation holds for  $t \ll 0$  and also when  $\alpha$  is order reversing. Consequently, each commensuration of  $F$  must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine  $\beta \in P$  induces a commensuration of  $F$  by conjugation. Suppose  $\beta(t+p) = \beta(t) + q$  for  $t \gg 0$  and  $\beta(t+p') = \beta(t) + q'$  for  $t \ll 0$ , with  $p, q, p', q' \in \mathbb{Z} \setminus \{0\}$ . Let  $U = [p', p]$  if  $\beta$  is

order preserving and set  $U = [p, p']$  otherwise. Then for  $f \in U$ , we have

$$f^\beta(t) = \begin{cases} \beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\ \beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0 \end{cases}$$

where  $k, k' \in \mathbb{Z}$  depend on  $f$ . Together with a similar argument for  $\beta^{-1}$  one easily sees that  $U^\beta = [q', q]$  or  $[q, q']$ , depending on whether  $\beta$  is order preserving or not. Theorem 3.1 is thus established.  $\square$

We immediately obtain the following corollaries from this result.

**Corollary 3.3** *A subgroup  $U$  of  $F$  of finite index is isomorphic to  $F$  if and only if  $U = [p, q]$  for some positive integers  $p$  and  $q$ .*

**Proof** Suppose  $U$  is a subgroup of finite index in  $F$ . If  $U$  is isomorphic to  $F$ , then there exists an eventually integrally periodically affine  $\alpha \in P$  with  $F^\alpha = U$  and calculations as above show that  $U$  must be of the form  $[p, q]$ . On the other hand, the final paragraph of the proof of the theorem read with  $p = p' = 1$  shows that  $[q', q]$  is isomorphic to  $F$  for every choice of positive integers  $q$  and  $q'$ . This completes the proof.  $\square$

Finally, since each subgroup of finite index in  $F$  contains  $[p, q]$  for some positive integers  $p$  and  $q$  by Proposition 2.1, we have the following results.

**Corollary 3.4** *Every finite-index subgroup of  $F$  is virtually  $F$ .*

**Corollary 3.5** *A group is commensurable with  $F$  if and only if it is a finite extension of  $F$ .*

## 4 The Structure of $\text{Com}(F)$

Descriptions of elements of  $\text{Com}(F)$  as conjugations in  $P$  allow us to study its structure as a group. An element  $\alpha$  of  $\text{Com}(F)$  is eventually integrally periodically affine, so there exist positive integers  $p, p', q, q'$  and a real number  $M$  such that

$$\begin{aligned} \alpha(t + p) &= \alpha(t) + q, \text{ for } t > M \\ \alpha(t + p') &= \alpha(t) + q', \text{ for } t < -M. \end{aligned}$$

We need a lemma about affine functions, whose proof is elementary and left to the reader.

**Lemma 4.1** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be an integrally periodically affine map, and assume that there are integers  $i, i', j, j'$  such that for all  $t \in \mathbb{R}$  we have*

$$f(t + i) = f(t) + j \quad \text{and} \quad f(t + i') = f(t) + j'.$$

Then we have

$$f(t + r) = f(t) + s,$$

where  $r = \gcd(i, i')$  and  $s = \gcd(j, j')$ .

Furthermore, we have

$$\frac{i}{j} = \frac{i'}{j'}.$$

From this lemma, we see that the integers  $p, p', q, q'$  for element of  $\text{Com}(F)$  depend only on the element.

We recall that  $\text{Com}(F)$  has a subgroup of index 2, denoted  $\text{Com}^+(F)$ , formed by the commensurations induced by conjugations by piecewise-linear maps which preserve the orientation of  $\mathbb{R}$ .

**Proposition 4.2** *There exists a surjective homomorphism  $\Phi: \text{Com}^+(F) \rightarrow \mathbb{Q}^* \times \mathbb{Q}^*$  defined by*

$$\Phi(f) = \left( \frac{p}{q}, \frac{p'}{q'} \right).$$

Here  $\mathbb{Q}^*$  denotes the multiplicative group of the positive rational numbers.

The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map capture the behavior at both ends, eventually near  $-\infty$  and eventually near  $+\infty$ . The two numbers  $p/q$  and  $p'/q'$  measure the "rate of growth" of the map at both ends.

A corollary of this result is that, as expected,  $\text{Com}(F)$  is infinitely generated.

## 5 Commensurations as Quasi-isometries

Let  $G$  be a finitely generated group. Quasi-isometries of  $G$  can be naturally composed, and there is a natural notion of equivalence class of quasi-isometries. Two quasi-isometries are considered equivalent if they are a bounded distance apart in the sense that  $f$  and  $g$  are considered equivalent if there exists a number  $M > 0$  such that  $d(f(t), g(t)) \leq M$  for all  $t$  in  $G$ .

Equivalence classes of quasi-isometries form elements of the group of quasi-isometries  $QI(G)$  of  $G$ . It is well known that the commensurator group admits a map to the quasi-isometry group, since all commensurations give maps between finite index subgroups which are canonically quasi-isometric to the ambient group. The result we want to prove in this section is that for Thompson's group  $F$ , this map is one-to-one.

**Theorem 5.1** *The natural homomorphism  $\text{Com}(F) \rightarrow QI(F)$  is injective.*

We begin with an elementary lemma.

**Lemma 5.2** *Given an element  $\tau \in P$  which is different from the identity, there exist two intervals  $I$  and  $J$  of the real line, whose endpoints are dyadic integers, with  $\tau(I) = J$ , and such that  $I \cap J = \emptyset$ .*

**Proof** The case when the slope of  $\tau$  is always 1 or  $-1$  is trivial. For a map  $t \mapsto t + k$  has a small interval (of length less than  $k$ ) whose image is disjoint from it. If  $\tau = -Id$  the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope  $\pm 2^i$  with  $i \neq 0$ . Assume without loss of generality (by possibly taking  $\tau^{-1}$  instead of  $\tau$ ) that  $i > 0$ . Hence there are two intervals  $[a, b]$  and  $[c, d]$  such that  $\tau(a) = c$  and  $\tau(b) = d$  and also  $d - c = 2^i(b - a)$ . It is possible that  $[a, b]$  and  $[c, d]$  overlap, but since  $[c, d]$  is much larger than  $[a, b]$  (at least twice the size), we can choose as  $J$  a small interval inside  $[c, d]$  which is disjoint from  $[a, b]$ . By construction, the preimage  $I$  of  $J$  is in  $[a, b]$ , and hence  $I$  and  $J$  are disjoint.  $\square$

**Proof of Theorem 5.1** We now take a nontrivial  $\tau \in \text{Com}(F)$ . By the previous lemma, there exist intervals  $I$  and  $J$  satisfying the conditions stated above and, in addition, that  $I$ , and hence  $J$ , have endpoints of the form  $k/2^j$  and  $(k+1)/2^j$ . We consider all elements of  $F$  whose support (that is, the part where they are not the identity) is contained in  $I$ . Those elements form a subgroup which is isomorphic to  $F$  itself. Let  $f$  be one such element. Since its support is inside  $I$ , its image under the commensuration  $\tau$ , that is,  $f^\tau = \tau \circ f \circ \tau^{-1}$ , has support inside  $J$ .

Hence, the distance (inside  $F$ ) from  $f$  to  $f^\tau$  is given by the distance from the identity to the element  $f^\tau f^{-1}$ . But this element has its support inside the disjoint union  $I \cup J$ , and the two parts are independent from each other (one given by  $f$  and the other one by  $f^\tau$ ). By work of Cleary and Taback [5], this subgroup—elements with support in  $I \cup J$  which is a direct product of two clone subgroups in their terminology—is

