IOC 17013 — Preliminary discussion for Lecture 6

Consider the standard feedback arrangement with $F(s) = 1$. One has

$$y = Tr + SPd - Tn,$$

with

$$S = \frac{1}{1+L}, \quad T = 1 - S = \frac{L}{1+L}, \quad L = PC.$$

1. **Disturbance rejection.** We want SP small at the disturbance frequency band $[0, \omega_{\mathrm{sy}}]$. Assuming P is not small, this implies that S must be small in $[0, \omega_{\mathrm{sy}}]$ and hence L large in $[0, \omega_{\mathrm{sy}}]$.

2. **Signal tracking.** We want $T \sim 1$ in $[0, \omega_{\mathrm{r}}]$. This means S small in $[0, \omega_{\mathrm{r}}]$ and hence L large in $[0, \omega_{\mathrm{r}}]$.

3. **Noise attenuation.** We want T small in $[\omega_{\mathrm{ty}}, \infty]$, and hence L small in $[\omega_{\mathrm{ty}}, \infty]$.

Typically, ω_{sy}, ω_{r} are small and ω_{ty} is large, so the above requirements are not in conflict.

Condition 2 can be related to the quantification of the relative error at a given frequency. Assume $||SW_1||_\infty < 1$, which implies

$$|S(j\omega)W_1(j\omega)| < 1 \quad \forall \omega.$$

If $r(t) = Ae^{j\omega t}$, one has $e(t) = S(j\omega)Ae^{j\omega t}$ and

$$\frac{|e(t)|}{A} = |S(j\omega)| < \frac{1}{|W_1(j\omega)|}.$$

If

$$|W_1(j\omega)| = \begin{cases} \sim k & \omega \in [0, \omega_{\mathrm{r}}], \\ \to 0 & \text{for } \omega \gg \omega_{\mathrm{r}}, \end{cases}$$

then

$$\frac{|e(t)|}{A} < \frac{1}{k} \quad \text{for } \omega \in [0, \omega_{\mathrm{r}}],$$

while no bound on the relative error is imposed for $\omega \gg \omega_{\mathrm{r}}$. Hence,

$$|S(j\omega)| < \frac{1}{k} \quad \text{in } [0, \omega_{\mathrm{r}}]$$

yields a relative tracking error of $1/k$ at the requested frequency band. In terms of L

$$\frac{1}{|1 + L(j\omega)|} < \frac{1}{k} \quad \Rightarrow \quad k < |1 + L(j\omega)|.$$

Since, as desired, $k \gg 1$, $|1 + L(j\omega)|$ must be large and 1 can be disregarded, yielding the approximate condition

$$|L(j\omega)| > k \gg 1 \quad \omega \in [0, \omega_{\mathrm{r}}].$$

Hence, a high gain is necessary at the tracking frequency band. Notice that this also helps to reject disturbances in that band.

Likewise, condition 3 can be related to the robust stability criterion $||W_2 T||_\infty < 1$. Normally, W_2 is small at low frequencies and $|W_2(j\omega)| \to K \gg 1$ at high frequencies, since uncertainty goes up with frequency. Hence

$$|T(j\omega)| < \frac{1}{K} \ll 1$$

at high frequencies, which is also in the line of the noise attenuation requirement. In terms of L this is

$$\left|\frac{L(j\omega)}{1 + L(j\omega)}\right| < \frac{1}{K} \quad \text{for } \omega \text{ large.}$$

Thus

$$K < \frac{|1 + L(j\omega)|}{|L(j\omega)|} \sim \frac{1}{|L(j\omega)|},$$

because $|L(j\omega)|$ can be disregarded in front of 1 if the above is to hold for $K \gg 1$. Finally

$$|L(j\omega)| < \frac{1}{K} \ll 1 \quad \text{for } \omega \text{ large.}$$

A small gain at high frequencies is thus necessary for both robust stability and noise attenuation.