Lecture 2

Basic concepts of feedback control

This lecture follows Chapter 3 of Doyle-Francis-Tannenbaum

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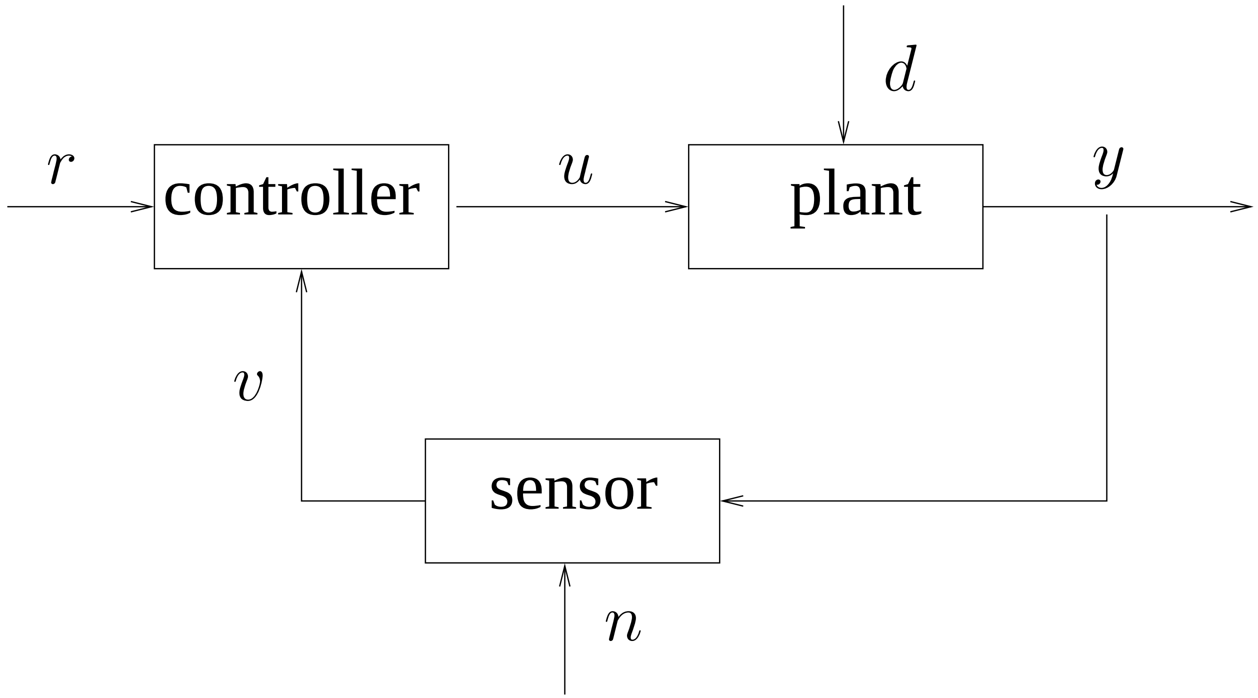
Basic feedback loop (I)



r reference or command input

v sensor output

u actuating signal, plant input

d external disturbance

y plant output and measured signal

n sensor noise

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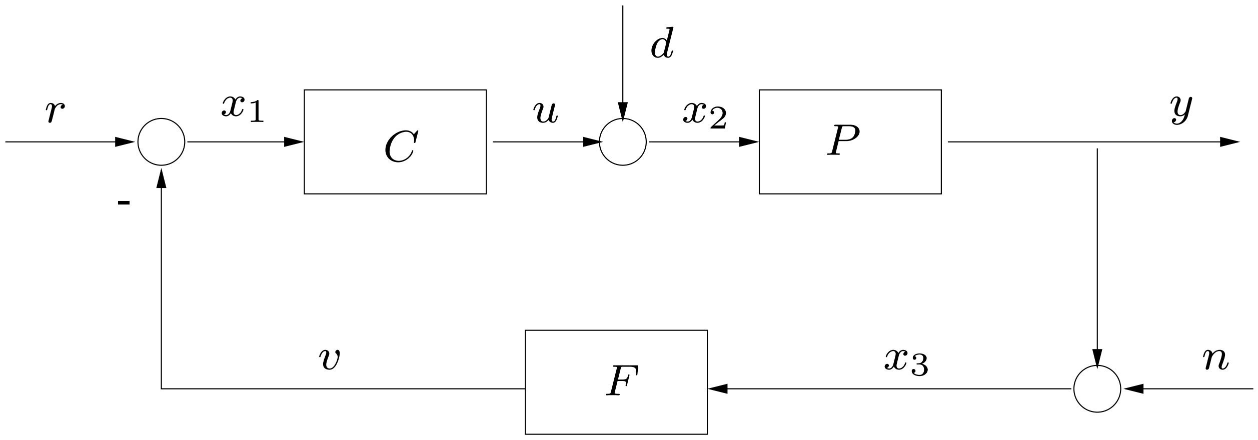
Basic feedback loop (II)

We assume that the outputs are linear functions of the sum/difference of the inputs:

$$y = P \cdot (d + u) \quad v = F \cdot (y + n) \quad u = C \cdot (r - v)$$



In matrix form

$$\begin{pmatrix} 1 & 0 & F \\ -C & 1 & 0 \\ 0 & -P & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

Basic feedback loop (III)

- The system is **well-posed** iff the three internal signals x_1, x_2, x_3 (and hence y, u and v) can be expressed in terms of the three inputs r, d and n.
- Well-posedness is equivalent to $1 + PCF \not\equiv 0$.
- A system that is not well-posed constraints the inputs, which makes no sense.
- If $P = s$, $C = -\frac{1}{s}$ and $F = 1$ we have

$$
\begin{aligned}
x_1 &= r - x_3, \\
\dot{x}_2 &= \dot{d} - x_1, \\
x_3 &= n + \dot{x}_2,
\end{aligned}
$$

from which $r = \dot{d} + n$.

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Basic feedback loop (IV)

- For a well-posed system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1+PCF} \begin{pmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

- A system is **strongly well-posed** if, for P, C and F proper, the above nine transfer functions are proper.
- A system is strongly well-posed iff $1 + PCF$ is not strictly proper, *i.e.*

$$PCF(\infty) \neq -1.$$

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Basic feedback loop (V)

- For any *physical* LTI system, the transfer function is strictly proper: the system cannot react to an input of ever-increasing frequency.
- However, for inputs of very high frequency the system may become nonlinear (excitation of nonlinear dynamics).
- In chemical engineering, PID controllers are common

$$k_1 + \frac{k_2}{s} + k_3 s.$$

This can be approximated over any frequency range by a proper controller

$$k_1 + \frac{k_2}{s} + \frac{k_3 s}{\tau s + 1}.$$

- For most of the course, we will assume that

$$\boxed{P \text{ is strictly proper, } C \text{ and } F \text{ are proper}}$$

This implies strong well-posedness.

- In the cases when P is only proper, we will assume that $|PCF|(\infty) < 1$, which again ensures strong well-posedness.

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Internal stability (I)

- BIBO systems. Consider a LTI SISO system with a proper and stable transfer function H:

$$H(s) = H_0 + H_1(s),$$

with H_0 a constant, H_1 strictly proper and all poles of H_1 with $\Re(s) < 0$. In the time domain

$$y(t) = H_0 u(t) + \int_{-\infty}^{+\infty} h_1(t-\tau) u(\tau)\, \mathrm{d}\tau.$$

If the input is bounded, *i.e.* $|u(t)| \leq c\ \forall t$,

$$|y(t)| \leq |H_0| c + c \int_{-\infty}^{+\infty} |h_1(\tau)|\, \mathrm{d}\tau.$$

For stable systems, $h_1 \in L_1(\mathbb{R})$ and the left-hand side is finite. Hence, the output is bounded for all t.

- A feedback system is said to be **internally stable** if the nine transfer functions are stable. Then the internal signals are bounded whenever the three external inputs are bounded.

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Internal stability (II)

- Write P, C, F as ratios of coprime polynomials (no common factors):

$$P = \frac{N_P}{M_P}, \quad C = \frac{N_C}{M_C}, \quad F = \frac{N_F}{M_F}.$$

- The **characteristic polynomial of the feedback system** is

$$N_P N_C N_F + M_P M_C M_F.$$

- The **closed-loop poles** of the feedback system are the zeros of its characteristic polynomial.

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Internal stability (III)

- **First theorem of internal stability.** The feedback system is internally stable iff there are no closed-loop poles in $\Re(s) \geq 0$.
- **Second theorem of internal stability.** The feedback system is internally stable iff the following two conditions hold:
  - The transfer function $1 + PCF$ has no zeros in $\Re(s) \geq 0$.
  - There is no pole-zero cancellation in $\Re(s) \geq 0$ when the product PCF is formed.

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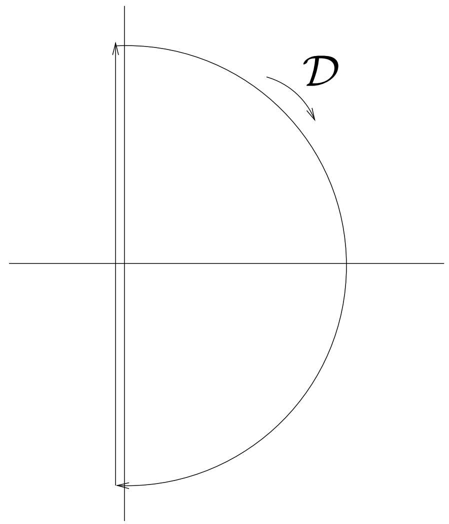
Internal stability (IV)

Consider the curve $\mathcal{D}$ in the complex plane formed by going up the imaginary axis and around the right half-plane following a semicircle of infinite radius. The **Nyquist plot** of H is the curve in the complex plane traced by the point $H(s)$ as s follows $\mathcal{D}$.



Nyquist's criterion: Construct the Nyquist plot of PCF, indexing to the left around any pole in the imaginary axis. Let n denote the total number of poles of P, C and F in $\Re(s) \geq 0$. Then the feedback system is internally stable iff the Nyquist plot does not pass through the point -1 and encircles it exactly n times counterclockwise.

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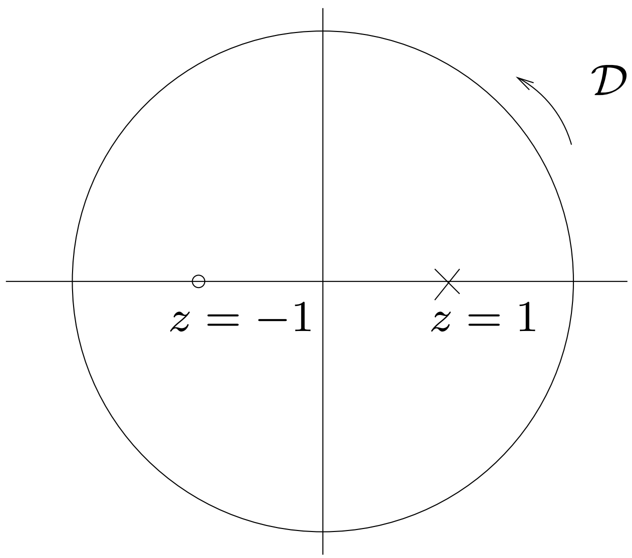
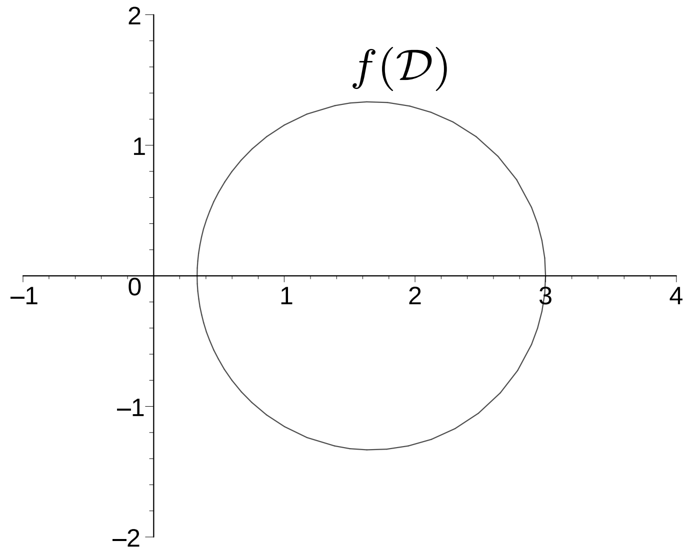
Internal stability (V)

Proof of Nyquist's criterion

- **The argument principle.** Let $f(z)$ be a complex function and let $\mathcal{D}$ be a closed curve in the complex plane. Assume that f is analytic on $\mathcal{D}$ and has N zeros and P poles (counting multiplicity in both cases) inside the region closed by $\mathcal{D}$. Then, as z tracks $\mathcal{D}$ counterclockwise, $f(z)$ encircles the origin exactly $N - P$ times counterclockwise.
- Consider $f(z) = (z+1)/(z-1)$ and let $\mathcal{D}$ be the circle of radius 2 and center the origin. It contains one zero and one pole of f, so $N - P = 1 - 1 = 0$. With $z = 2e^{j\theta}$ one has $f(z) = \frac{3}{5-4\cos\theta} - j\frac{4\sin\theta}{5-4\cos\theta}$ and





Internal stability (VI)

Proof of Nyquist's criterion (cont'd)

- Nyquist's criterion can be derived from the second theorem of internal stability.
- Since there is no zero-pole cancellation when forming PCF, the poles of $1 + PCF$ are the poles of P, C and F.
- Consider the Nyquist plot of $1 + PCF$. Since $\mathcal{D}$ contains all of the region $\Re(s) \geq 0$ and we do not want any zero there, the argument principle tells us that the Nyquist plot of $1 + PCF$ encircles the origin n times counterclockwise (notice that $\mathcal{D}$ is oriented clockwise). Hence, the Nyquist plot of PCF encircles the point $z = -1$ exactly n times counterclockwise.

Asymptotic tracking (I)

- For the rest of the lecture we consider a feedback system with $F = 1$.
- Suppose that $n = d = 0$. The tracking error is $e = r - y$ and the transfer function from r to e is the **sensitivity function**

$$S = \frac{1}{1 + L}$$

  where $L = PC$ is the **loop transfer function**.
- We want to study the system's capability to track certain test inputs asymptotically:
  - the **step input** of height c, $r(t) = c\theta(t)$,
  - the **ramp input** with slope c , $r(t) = ct\theta(t)$.

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Asymptotic tracking (II)

Asymptotic tracking theorem

- Assume that the feedback system is internally stable and that $n = d = 0$. Then
  - for a step input, $e(t) \to 0$ as $t \to +\infty$ iff S has at least one zero at the origin.
  - for a ramp input, $e(t) \to 0$ as $t \to +\infty$ iff S has at least two zeros at the origin.
- Consider for instance a system with a plant $P(s) = 1/s$ (pure integrator). Then $S(s) = \frac{1}{1+\frac{1}{s}C(s)} = \frac{s}{s+C(s)}$. For the system to be internally stable, we must choose $C(s)$ so that it has no zero is $s = 0$ (to avoid cancellation with the pole of P at $s = 0$), and that $1 + PC = s + C(s)$ has no zero in $\Re(s) \geq 0$. If $C(s) = 1$ the system will be able to track a step, while if $C(s) = (s+1)/s$ it will be able to track both a step and a ramp.

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Asymptotic tracking (III)

- **Final value theorem**. If $X(s)$ is a rational Laplace transform that has no poles in $\Re(s) \geq 0$, except possibly a simple pole in $s = 0$, then $x(t)$ has an asymptotic value and

$$\lim_{t \to +\infty} x(t) = \lim_{s \to 0} sX(s).$$

- **Proof of asymptotic tracking theorem**. Let's consider a step input, with Laplace transform $R(s) = c/s$. Since S is the transfer function from r to e, we have $E(s) = S(s)\frac{c}{s}$. We can use the final value theorem because S has no pole with $\Re(s) \geq 0$ (otherwise the system wouldn't be internally stable), and hence E only has the unstable pole at $s = 0$ coming from the input. Then

$$\lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} cS(s) = cS(0) = 0$$

iff S has at least a zero at the origin. Similarly for a ramp input, with $R(s) = c/s^2$.

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Performance (I)

- Instead of perfect tracking of a single test signal, here we want to study asymptotically bounded error tracking of a set of signals.
- Again let $L = PC$ be the loop transfer function and $S = 1/(1+L)$ the sensitivity function. We will assume S to be stable. Since P is strictly proper, $S(j\infty) = 1$.
- The transfer function from r to y is $T = PC/(1+PC)$. Assume we have an uncertainty in the plant, $P \to P + \Delta P$ and let's compute the ratio of the relative variation of T and the relative variation of P (this will give a clue of the sensitivity of the tracking to variations in the plant):

$$\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{\mathrm{d}T}{\mathrm{d}P}\frac{P}{T} = \frac{1}{1+PC} = S.$$

T is called the **complementary sensitivity function**.

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Performance (II)

- Consider a system such that the only thing we know about the command input is that it is a sinusoid of amplitude no greater than one, *i.e.*

$$r(t) = Ae^{j\omega t}, \ |A| \leq 1 \text{ unknown}, \ \omega \text{ unknown}.$$

- In absence of any other inputs, $e(t) = S(j\omega)Ae^{j\omega t}$ and

$$|e(t)| = |S(j\omega)||A| \leq |S(j\omega)| \leq ||S||_\infty.$$

- If we want $|e(t)| < \epsilon$ for all t, we can impose $||S||_\infty < \epsilon$. Introducing the (trivial) **weighting function**

$$W_1(s) = \frac{1}{\epsilon},$$

this is

$$||W_1 S||_\infty < 1.$$

Performance (III)

- Consider now that r is a *prefiltered* input, with a filtering transfer function W_1, *i.e*, in the transform domain,

  $$R(s) = W_1(s) R_{pf}(s),$$

  where $r_{pf}(t)$ is again a sinusoid of amplitude not greater than one.

- We have now $e(t) = W_1(j\omega) S(j\omega) A e^{j\omega t}$ and

  $$|e(t)| \leq |W_1(j\omega) S(j\omega)| \leq ||W_1 S||_\infty,$$

  so, again, a bound on $|e(t)|$ for all t translates to a bound for $||W_1 S||_\infty$.

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Performance (IV)

- Remember that, from Parseval's theorem, $||y||_2^2 = ||Y||_2^2$.
- Consider again a prefiltered input, but now we only know about it that its 2-norm, $||r_{pf}||_2$, is not greater than one.
- Assume that we want to measure the tracking error by its $2-$norm, $||e||_2$.
- Remembering that the 2-norm/2-norm system gain is the ∞-norm of the transfer function, we write

$$||e||_2 \leq ||W_1 S||_\infty ||R_{pf}||_2 \leq ||W_1 S||_\infty,$$

where we have used $||R_{pf}||_2 = ||r_{pf}||_2 \leq 1$ and $E(s) = S(s)R(s) = S(s)W_1(s)R_{pf}(s)$.

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Performance (V)

- In certain problems, experienced engineers know that the controller performs adequately if $S(j\omega)$ is below a given curve for all the frequencies. This can be written as

$$|S(j\omega)| < |W_1(j\omega)|^{-1} \ \ \forall\omega$$

for a suitable W_1, *i.e.* $|S(j\omega)W_1(j\omega)| < 1 \ \forall\omega$, and again $||W_1 S||_\infty < 1$.

As the above examples show, many performance problems can be expressed as bounds on the ∞-norm of S weighted by a function W_1:

$$\boxed{||W_1 S||_\infty < 1}$$

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