

# Lecture 3

## *Uncertainty and robustness*

This lecture follows Chapter 4 of Doyle-Francis-Tannenbaum, with proofs and some discussions omitted

# Plant uncertainty (I)

- We model uncertainty of the plant using a set  $\mathcal{P}$  of plants. The only thing we know about the plant is that it, or rather its transfer function, belongs to  $\mathcal{P}$ .
- **Structured uncertainty:**  $P(s)$  is parameterized by a finite number of scalars.
- Consider a *RCL* series circuit, with transfer function (from input voltage to capacitor voltage)

$$P(s) = \frac{1}{LCs^2 + RCs + 1}$$

with (SI units)  $L = 1$ ,  $C = 2$ , and such we do not know  $R$  exactly, only that  $1 \leq R \leq 5$ . The set of plants is then

$$\mathcal{P} = \left\{ \frac{1}{2s^2 + as + 1}; 2 \leq a \leq 10 \right\}.$$

- Another example of structured uncertainty is that of a **discrete set of plants:**

$$\mathcal{P} = \{P_1(s), P_2(s), \dots\}.$$

# Plant uncertainty (II)

- **Unstructured uncertainty** is more important:
  - it can cover unmodeled dynamics.
  - it allows nice analytical results.
- Basic model: **multiplicative disk uncertainty**.
- Any plant in the set is of the form

$$\tilde{P} = (1 + \Delta W_2)P$$

where

- $P$  is the nominal plant.
- $W_2$ , the weight, is a fixed stable transfer function.
- $\Delta$  is a variable stable transfer function, with  $\|\Delta\|_\infty \leq 1$ .

It is assumed that no unstable poles of  $P$  are canceled in forming  $\tilde{P}$ .

# Plant uncertainty (III)

- $\Delta W_2$  is the normalized plant perturbation:

$$\frac{\tilde{P} - P}{P} = \Delta W_2.$$

- If  $\|\Delta\|_\infty \leq 1$  then

$$\left| \frac{\tilde{P}(j\omega)}{P(j\omega)} - 1 \right| \leq |W_2(j\omega)|, \quad \forall \omega.$$

- For each frequency, the point  $\tilde{P}/P$  lies in the disk with center 1 and radius  $|W_2|$ . Hence the name.
- Typically,  $|W_2(j\omega)|$  is an increasing function of  $\omega$  (plant uncertainty increases at higher frequencies).  $\Delta$  allows for phase uncertainty and acts as a scaling factor.
- How does one get the weighting function in specific cases?

# Plant uncertainty (IV)

**A double integrator with time delay.** Consider the nominal plant

$$P(s) = \frac{1}{s^2}.$$

Suppose that a more detailed analysis shows that the real plant has an unknown time-delay

$$\tilde{P}(s) = e^{-\tau s} \frac{1}{s^2}, \quad 0 \leq \tau \leq 0.1.$$

Since  $\frac{\tilde{P}(j\omega)}{P(j\omega)} = e^{-\tau j\omega}$ , we have to choose  $W_2$  such that

$$|e^{-\tau j\omega} - 1| \leq |W_2(j\omega)|, \quad \forall \omega, \quad \forall \tau \in [0, 0.1].$$

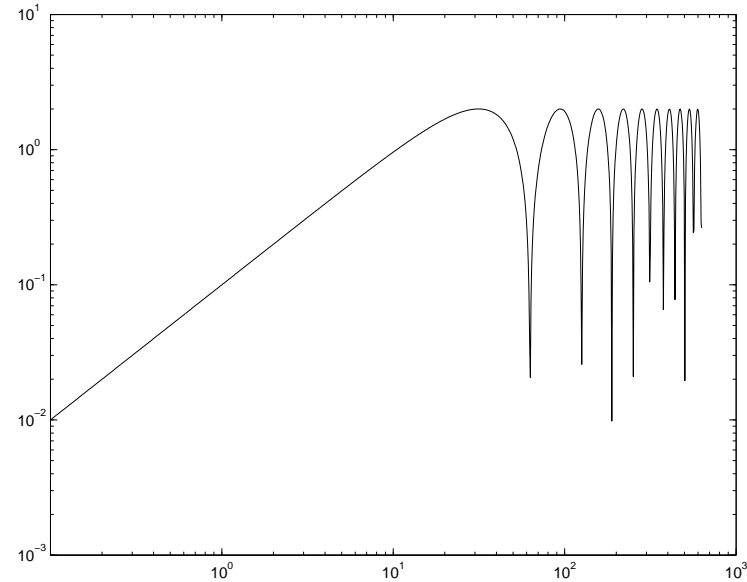
Clearly, we can work with the worst case value  $\tau = 0.1$ .

# Plant uncertainty (V)

The Bode plot of

$$e^{-0.1s} - 1$$

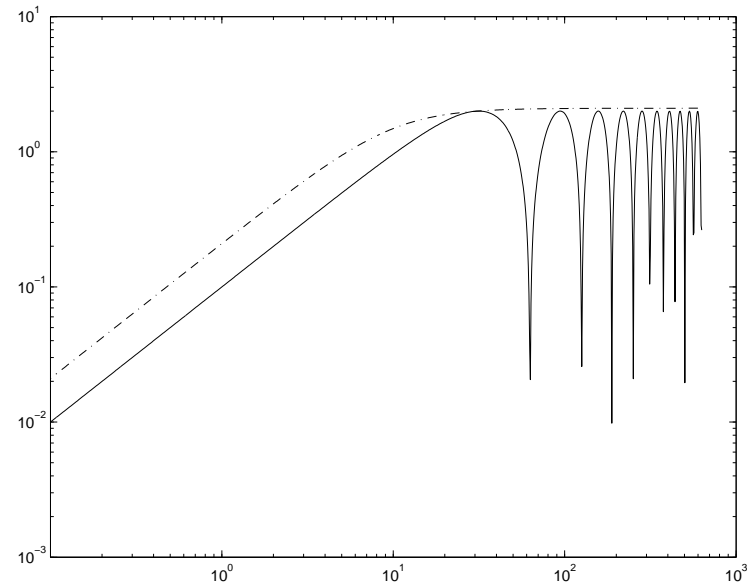
increases linearly and then can be enveloped  
by a constant  $\sim 2.1$ .



A little experimenting yields

$$W_2(s) = \frac{0.21s}{0.1s + 1}$$

whose Bode plot is given by the dot-dash  
line.



# Plant uncertainty (VI)

**An unstable plant with unknown gain.** Consider the set of plants given by

$$\tilde{P}(s) = \frac{k}{s-2}, \quad 0.1 \leq k \leq 10.$$

We will embed it in a family with nominal plant  $P(s) = \frac{k_0}{s-2}$  for a suitable  $k_0 \in [0.1, 10]$ .

Since  $\tilde{P}(j\omega)/P(j\omega) = k/k_0$ , we must demand that

$$\left| \frac{k}{k_0} - 1 \right| \leq |W_2(j\omega)|, \quad \forall \omega, \forall k \in [0.1, 10].$$

Since the left-hand side does not depend on  $\omega$ , we will be able to choose  $W_2$  constant. One can see that the maximum of the left-hand side for  $k \in [0.1, 10]$  is minimized for  $k_0 = 5.05$ , and it takes the value  $4.95/5.05$ . Hence we can represent the set with the nominal plant

$$P(s) = \frac{5.05}{s-2}$$

and the weight  $W_2(s) = 4.95/5.05$ . We could also have fixed  $k_0$  from the beginning, but then we would have got a bigger  $W_2$ .

# Plant uncertainty (VII)

- Generally, the multiplicative uncertainty disk set is bigger than the original uncertainty model.
- Controllers designed for the bigger set may not be as good as controllers designed for the original model (if they could be computed) when applied to the original uncertainty set.
- **Uncertainty disk plant models yield conservative controllers.** However, they are computable, which can be impossible for the original model.
- Other uncertainty disk models:

$$P + \Delta W_2 \quad (\text{additive}), \quad \frac{P}{1 + \Delta W_2 P} \quad (\text{feedback}).$$

# Plant uncertainty (VIII)

**Feedback disk uncertainty model.** Let the set be given by

$$\frac{1}{s^2 + as + 1}, \quad 0.4 \leq a \leq 0.8.$$

We can write  $a = 0.6 + 0.2\Delta$  with  $-1 \leq \Delta \leq 1$ .

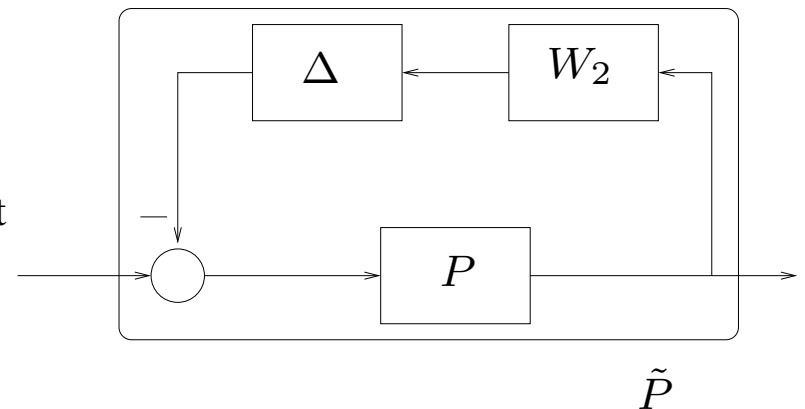
Then the family can be described by

$$\frac{P(s)}{1 + \Delta W_2(s)P(s)}, \quad -1 \leq \Delta \leq 1,$$

with the weight  $W_2(s) = 0.2s$  and the nominal plant

$$P(s) = \frac{1}{s^2 + 0.6s + 1}.$$

This can be interpreted as a feedback around the nominal plant.



# Robust stability (I)

- Suppose that  $P \in \mathcal{P}$ , and, given a controller  $C$ , consider a characteristic of the feedback system.
- The controller  $C$  is **robust** with respect this characteristic if this characteristic holds for every plant in  $\mathcal{P}$ .
- Hence, a controller  $C$  provides **robust stability** if it provides internal stability for every plant in  $\mathcal{P}$ .
- We would like to have a test for robust stability, involving only  $C$  and the set  $\mathcal{P}$ .
- Alternatively, given  $C$  and a measure of the size of  $\mathcal{P}$ , we would like to have a bound on the maximum size of  $\mathcal{P}$  such that  $C$  internally stabilizes all of  $\mathcal{P}$ .

# Robust stability (II)

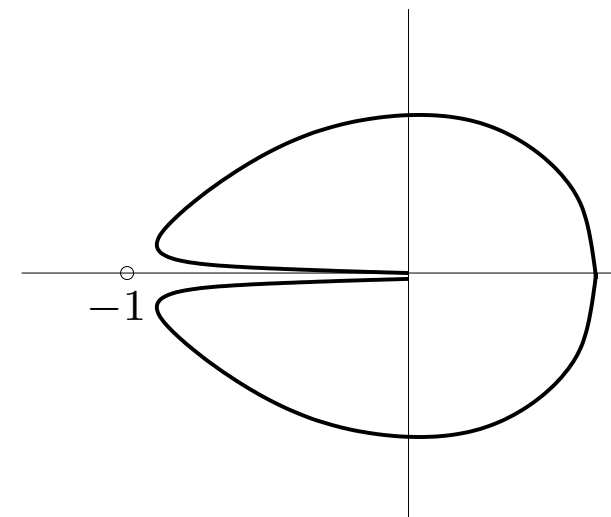
**Classical measures of stability margin.** Assume that the feedback system is internally stable with nominal plant  $P$  and controller  $C$ .

- Perturb the plant to  $kP$ ,  $k > 0$ . The **upper gain margin**,  $k_{\max}$ , is the first value of  $k$  greater than one such that the feedback system is not internally stable. If there is no such value then  $k_{\max} = \infty$ . On the Nyquist plot of  $L = PC$ , the point on  $(-1, 0)$  where the plot intersects the negative real axis is given by  $-1/k_{\max}$ . This follows from the fact that the Nyquist plot of  $kL$  is just a dilatation of that of  $L$ . When the plot hits  $-1$ , the system becomes unstable and then remains so since it has an extra encirclement.

# Robust stability (III)

- Perturb the plant to  $e^{-j\phi}P$ . The **phase margin**,  $\phi_{\max}$ , is the maximum number such that internal stability holds for  $0 \leq \phi \leq \phi_{\max}$ . If the system is internally stable for all  $\phi$ , we say that the margin is  $\infty$ . Since the Nyquist plot of  $e^{-j\phi}L$  is that of  $L$  rotated  $\phi$  clockwise,  $\phi_{\max}$  is the angle through which the Nyquist plot of  $L$  must be rotated until it passes through the critical point  $-1$ .

If either of the margins is small, the system is close to instability. However, both margins can be large and the system can still be close to instability, which can be reached with simultaneous variations in gain and phase. The system to the right has both margins equal to infinity, yet it is on the verge of instability.



# Robust stability (IV)

Now we turn to an analytical test of robust stability for the multiplicative uncertainty model, in terms of the complementary sensitivity function  $T$  and the weight  $W_2$ . We assume that  $C$  provides internal stability for the nominal plant  $P$ .

**Theorem 1 (robust stability theorem).** For the multiplicative uncertainty model,  $C$  provides robust stability iff  $\|W_2T\|_\infty < 1$ .

**Sufficiency proof.** We will use the second theorem of internal stability. First of all, due to the constraints on  $\Delta$  and  $W_2$ , there are no unstable zero/pole cancelations when forming  $\tilde{P}C$ , since  $C$  provides internal stability for the nominal plant and no such cancelation happens for  $PC$ . We have  $\tilde{P}C = (1 + \Delta W_2)L$  and

$$1 + \tilde{P}C = 1 + (1 + \Delta W_2)L = (1 + L)(1 + \Delta W_2T).$$

Furthermore,

$$\|\Delta W_2T\|_\infty \leq \|\Delta\|_\infty \|W_2T\|_\infty \leq \|W_2T\|_\infty < 1.$$

# Robust stability (V)

Now we invoke the

**Maximum modulus theorem.** Let  $\Omega$  be a nonempty, open, connected set in  $\mathbb{C}$  and let  $H$  be analytic in  $\Omega$ . Suppose that  $H$  is not a constant. Then  $|H|$  does not attain a maximum value in  $\Omega$ .

We apply this to  $W_2T$  with  $\Omega$  the open right half plane, where  $W_2T$  is analytic (in fact, it is analytic on the closed right half plane). We conclude that  $|W_2T|(z)$  does not reach a maximum in  $\Omega$ , which implies that the values there are bounded by the values in the border, which in turn are bounded by  $\|W_2T\|_\infty$ , which is less than 1. Hence  $1 + \Delta(z)W_2(z)T(z)$  cannot be zero for any  $z$  with  $\Re(z) \geq 0$ . Since  $1 + L$  has no unstable zeros, the same applies to  $1 + \tilde{L}$ . This concludes the proof.

# Robust stability (VI)

- Assume  $C$  provides robust stability for the disk multiplicative model given by  $W_2$ . We may ask how much can we increase  $\|\Delta\|_\infty$  (which for the plants in the original set is no greater than 1) before we encounter instability. Denote by  $\beta_{\text{sup}}$  the least upper bound on  $\beta$  such that  $C$  achieves internal stability for all  $\|\Delta\|_\infty \leq \beta$ . Then  $\beta_{\text{sup}}$  is a stability margin with respect to this uncertainty model.
- **Proposition.** The stability margin for the disk multiplicative model is given by

$$\beta_{\text{sup}} = \frac{1}{\|W_2 T\|_\infty}.$$

Thus, the smaller  $\|W_2 T\|_\infty$  is, the safer is the controller.

# Robust stability (VII)

One can find robust stability tests for the other two disk perturbation models. Results are given in the following table.

Perturbation	Condition
$(1 + \Delta W_2)P$	$\ W_2 T\ _\infty < 1$
$P + \Delta W_2$	$\ W_2 C S\ _\infty < 1$
$\frac{P}{1 + \Delta W_2 P}$	$\ W_2 P S\ _\infty < 1$

# Robust performance (I)

- Recall that when the nominal feedback system is internally stable, the nominal performance condition is  $\|W_1 S\|_\infty < 1$  and the robust stability condition (multiplicative model) is  $\|W_2 T\|_\infty < 1$ .

- If  $P$  is perturbed to  $(1 + \Delta W_2)P$ ,  $S$  is perturbed to

$$\frac{1}{1 + (1 + \Delta W_2)L} = \frac{S}{1 + \Delta W_2 T}.$$

- The robust performance condition is then that both
  - $\|W_2 T\|_\infty < 1$ , to keep internal stability,
  - $\left\| \frac{W_1 S}{1 + \Delta W_2 T} \right\|_\infty < 1$  for all allowable  $\Delta$ , to keep performance,

hold. The second condition is awkward, due to the  $\Delta$  in the denominator.

# Robust performance (II)

The practical check is given by the following

**Theorem 2 (robust performance theorem).** A necessary and sufficient condition for robust performance for the disk multiplicative model with weight  $W_2$  and with reference signals described by  $W_1$  is

$$\| |W_1 S| + |W_2 T| \|_\infty < 1.$$

The proof is (more or less) simple algebra and the key observation that the condition in the theorem is equivalent to

$$\|W_2 T\|_\infty < 1 \quad \text{and} \quad \left\| \frac{W_1 S}{1 - |W_2 T|} \right\|_\infty < 1.$$

# Robust performance (III)

- Again, we can provide results for the other disk uncertainty models. For the additive model it is

$$\| |W_1 S| + |W_2 C S| \|_\infty < 1,$$

while the results for the feedback models are quite messy (however, the situation is reversed if nominal performance conditions based on  $T$  instead of  $S$  are used).

- If we impose nominal performance and robust stability with a safety factor of 2 for both,

$$\| |W_1 S| \|_\infty < \frac{1}{2}, \quad \| |W_2 T| \|_\infty < \frac{1}{2},$$

then

$$|W_1(j\omega)S(j\omega)| + |W_2(j\omega)T(j\omega)| < \frac{1}{2} + \frac{1}{2} = 1 \quad \forall \omega$$

so  $\| |W_1 S| + |W_2 T| \|_\infty < 1$  and we automatically get robust performance. However, it can be difficult to find a controller with this safety factor for both conditions.