Lecture 4

Stabilization

This lecture follows Chapter 5 of Doyle-Francis-Tannenbaum, with proofs and Section 5.3 omitted

Stable plants (I)

- We assume that the plant P is already stable and parameterize all C for which the feedback system is internally stable. We set $F = 1$.
- $\mathcal{Q} = \{$ all stable, proper, real-rational functions$\}$.
- **Theorem 1**. Assume that $P \in \mathcal{Q}$. Then the set of all controllers C for which the feedback system is internally stable equals
$$\left\{ \frac{Q}{1 - PQ}, \; Q \in \mathcal{Q} \right\}.$$
- The more interesting case, when P is not stable, will be considered later.

Stable plants (II)

Proof of theorem 1

- Suppose C achieves internal stability. Then the transfer function from r to u belongs to $\mathcal{Q}$ and is given by

$$Q = \frac{C}{1+PC}, \quad \text{from which} \quad C = \frac{Q}{1-PQ}$$

and this is of the form stated.

- Now suppose that $Q \in \mathcal{Q}$ and define the controller $C = \frac{Q}{1-PQ}$. Substituting this into the matrix of 9 transfer functions yields

$$\begin{pmatrix} 1-PQ & -P(1-PQ) & -(1-PQ) \\ Q & 1-PQ & -Q \\ PQ & P(1-PQ) & 1-PQ \end{pmatrix}$$

and all of them are clearly stable (since P and Q are stable).

Stable plants (III)

Example. Suppose we want to find an internally stabilizing controller so that y asymptotically tracks a ramp r for the plant

$$P(s) = \frac{1}{(s+1)(s+2)}.$$

The transfer function S from r to e must have at least two zeros at $s = 0$. Parameterize C as in the theorem and, in order to get the two zeros, consider $Q \in \mathcal{Q}$ with two free parameters, for instance

$$Q(s) = \frac{as+b}{s+1}.$$

One gets

$$S(s) = \frac{1}{1+P(s)C(s)} = \frac{1}{1+P(s)\frac{Q(s)}{1-P(s)Q(s)}} = \frac{s^3+4s^2+(5-a)s+(2-b)}{(s+1)^2(s+2)},$$

so $a = 5$ and $b = 2$ and then $C(s) = \frac{(5s+2)(s+1)(s+2)}{s^2(s+4)}$. Notice that it cancels the two poles of the plant and it has two poles at $s = 0$.

Coprime factorization (I)

- Suppose now that P is not necessarily stable. Write it as the ratio of coprime polynomials $P = \frac{N}{M}$.
- Euclid's algorithm allows us to get two other polynomials X and Y such that $NX + MY = 1$.
- If we set $C = X/Y$, then $NX + MY$ is the characteristic polynomial of the feedback system. From the above construction this equals 1 and the system is internally stable (since it has no zeros, and in particular no zeros with $\Re(s) \geq 0$).
- The problem with this is that Y may be zero, and even if not, C may not be proper.

Coprime factorization (II)

- For instance, if $P(s) = 1/s$, we can take $N(s) = 1$ and $M(s) = s$. One solution to $NX + MY = 1$ is then $X(s) = 1$ and $Y(s) = 0$, for which X/Y is undefined. Another solution is $X(s) = -s + 1$, $Y(s) = 1$, for which X/Y is not proper.
- The way out of this it to arrange that N, M, X and Y are all elements of $\mathcal{Q}$ instead of polynomials.
- Two functions N and M of $\mathcal{Q}$ are **coprime** if there exist two other functions X and Y in $\mathcal{Q}$ such that

$$NX + MY = 1.$$

- For this to hold, N and M cannot have common zeros, nor can they be simultaneously strictly proper, for, if s_0 is a common zero or $s_0 = \infty$, then

$$0 = N(s_0)X(s_0) + M(s_0)Y(s_0) \neq 1.$$

This is also a sufficient condition for coprimeness.

Coprime factorization (III)

- Let P be a real-rational transfer function. A representation of the form
$$P = \frac{N}{M}, \quad N, M \in \mathcal{Q}$$
where M and N are coprime is called a **coprime factorization** of P over $\mathcal{Q}$.
- Our goal is to provide a method that, given P, finds four functions in $\mathcal{Q}$ satisfying
$$P = \frac{N}{M}, \quad NX + MY = 1.$$
- The construction of N and M is easy, and contains some degree of arbitrariety. We illustrate it with an example.

Coprime factorization (IV)

- Let $P(s) = 1/(s-1)$. To write $P = N/M$ with N and M in $\mathcal{Q}$, simply divide the numerator and denominator by a common polynomial with no zeros in $\Re(s) \geq 0$.
- Take for instance $(s+1)^k$, with $k = 1, 2, \ldots$. We get

$$P(s) = \frac{N(s)}{M(s)}, \quad N(s) = \frac{1}{(s+1)^k}, \quad M(s) = \frac{s-1}{(s+1)^k}.$$

- Since N and M are to be coprime, they cannot be strictly proper at once. Nothing can be done for N, but if $k = 1$ then M is not strictly proper. Hence, a solution to our problem is

$$N(s) = \frac{1}{s+1}, \quad M(s) = \frac{s-1}{s+1}.$$

- How to get the other two functions X and Y? Euclid's algorithm, which will be used in the main procedure.

Coprime factorization (V)

Euclid's algorithm.

- It computes the greatest common divisor of two polynomials $n(s)$, $m(s)$. When they are coprime, it can be used to obtain $x(s)$, $y(s)$ satisfying $nx + my = 1$.
- Assume that the degree of m is greater than or equal to the degree of n. If not, interchange the polynomials. Then
  1. Divide n into m to get $m = nq_1 + r_1$ with degree$(r_1) <$ degree(n).
  2. Divide r_1 into n to get $n = r_1 q_2 + r_2$ with degree$(r_2) <$ degree(r_1).
  3. Divide r_2 into r_1 to get $r_1 = r_2 q_3 + r_3$ with degree$(r_3) <$ degree(r_2).
  4. Stop when you reach a constant remainder r_k.
- If $r_k = 0$, then the polynomials are not coprime and their GCD is r_{k-1}.
- If $r_k \neq 0$, the relations can be worked upwards to get

$$r_k = an + bm.$$

Dividing by r_k we get

$$1 = \frac{a}{r_k} n + \frac{b}{r_k} m$$

from which x and y are then identified.

Coprime factorization (VI)

Euclid's algorithm (example).

- Consider $m(s) = s^3 + s + 1$, $n(s) = s^2 + s$.
- One has
  - $s^3 + s + 1 = (s^2 + s)\underbrace{(s-1)}_{q_1} + \underbrace{(2s+1)}_{r_1}$
  - $s^2 + s = (2s+1)\underbrace{\left(\frac{1}{2}s + \frac{1}{4}\right)}_{q_2} \underbrace{-\frac{1}{4}}_{r_2}$
- Since $r_2 = -1/4 \neq 0$, the polynomials are coprime and then

$$\begin{aligned} -\frac{1}{4} &= (s^2+s) - (2s+1)\left(\frac{1}{2}s + \frac{1}{4}\right) \\ &= (s^2+s) - \left((s^3+s+1) - (s^2+s)(s-1)\right)\left(\frac{1}{2}s + \frac{1}{4}\right) \end{aligned}$$

Multiplying by -4 we finally get

$$1 = \underbrace{(2s+1)}_{y(s)}(s^3+s+1) + \underbrace{(-2s^2+s-3)}_{x(s)}(s^2+s).$$

Coprime factorization (VII)

Main procedure.

- Now we attack the coprime factorization over $\mathcal{Q}$.
- Given P
  1. If P is stable, set $N = P$, $M = 1$, $X = 0$, $Y = 1$ and that's it; else continue.
  2. Transform $P(s)$ to $\tilde{P}(\lambda)$ under the mapping
     $$s = \frac{1-\lambda}{\lambda}.$$
  3. Write $\tilde{P}$ as a ratio of coprime polynomials
     $$\tilde{P}(\lambda) = \frac{n(\lambda)}{m(\lambda)}.$$
  4. Using Euclid's algorithm, find polynomials $x(\lambda)$, $y(\lambda)$ such that $nx + my = 1$.
  5. Transform $n(\lambda)$, $m(\lambda)$, $x(\lambda)$, $y(\lambda)$ to $N(s)$, $M(s)$, $X(s)$ and $Y(s)$ with the inverse mapping
     $$\lambda = \frac{1}{s+1}.$$

Coprime factorization (VIII)

- The method works because under the mapping $\lambda = \frac{1}{s+1}$, any polynomial in λ yields a proper rational transfer function with all poles at $s = -1$, and hence it is stable. The mapping is not unique: it just has to preserve this condition.
- **Example.** If

$$P(s) = \frac{1}{(s-1)(s-2)}$$

we have

$$\tilde{P}(\lambda) = \frac{\lambda^2}{6\lambda^2 - 5\lambda + 1}, \quad n(\lambda) = \lambda^2, \quad m(\lambda) = 6\lambda^2 - 5\lambda + 1.$$

Euclid's algorithm yields $x(\lambda) = -30\lambda + 19$, $y(\lambda) = 5\lambda + 1$ and hence

$$X(s) = \frac{19s - 11}{s+1}, \quad Y(s) = \frac{s+6}{s+1}$$

from which the stabilizing controller is

$$C(s) = \frac{19s - 11}{s+6}.$$

Controller parametrization (I)

- We have presented
    - a method to get all the stabilizing controllers for an already stable plant.
    - a method to get a stabilizing controller for a general plant.

  Now we want to get all the stabilizing controllers for a general plant.

- **Theorem 2 (Youla-Kucera parametrization).** Let $P = N/M$ be a coprime factorization of P over $\mathcal{Q}$ and let X, Y in $\mathcal{Q}$ satisfying $NX + MY = 1$. Then the set of all controllers C for which the feedback system is internally stable is
$$\left\{ \frac{X + MQ}{Y - NQ},\ Q \in \mathcal{Q} \right\}.$$

- This result reduces to Theorem 1 for stable plants, since then it suffices to take $N = P$, $M = 1$, $X = 0$, $Y = 1$.

Controller parametrization (II)

- **Example.** For

$$P(s) = \frac{1}{(s-1)(s-2)}$$

we have already found

$$N(s) = \frac{1}{(s+1)^2}, \quad M(s) = \frac{(s-1)(s-2)}{(s+1)^2}$$

and

$$X(s) = \frac{19s-11}{s+1}, \quad Y(s) = \frac{s+6}{s+1}.$$

Now, any controller of the form

$$C(s) = \frac{X+MQ}{Y-NQ}(s) = \frac{(19+Q)s^2+(8-3Q)s+8s-11+2Q}{s^2+7s+6-Q}$$

with $Q \in \mathcal{Q}$ will internally stabilize the feedback. In particular, for $Q = 0$ we get the controller from the previous example.

Controller parametrization (III)

- The Youla-Kucera parametrization has the important property that all the closed-loop transfer functions are affine functions of Q. In particular

$$\begin{aligned} S &= M(Y - NQ), \\ T &= N(X + MQ). \end{aligned}$$

- Given the freedom in choosing Q, we will use the Youla-Kucera parametrization to impose additional properties on the controller. In particular, we will search stabilizing controllers that achieve one of the following: asymptotic (tracking) conditions, strong (*i.e.* C stable) stabilization or simultaneous stabilization of two plants.

Asymptotic properties (I)

We will pursue the example of the plant

$$P(s) = \frac{1}{(s-1)(s-2)}$$

to illustrate how to impose asymptotic properties on the stabilizing controller. The problem is to find a proper C so that, simultaneously,

1. the feedback system is internally stable.
2. the final value of y equals 1 when r is a step function and $d = n = 0$.
3. the final value of y equals zero when $r = n = 0$ and $d(t) = A \sin 10t + B \cos 10t$.

Asymptotic properties (II)

We have obtained already suitable rational transfer functions N, M, X and Y for this example, and we know that all the stabilizing controllers are of the form

$$C = \frac{X + MQ}{Y - NQ}.$$

For C of this form, the transfer function from r to y is $N(X + MQ)$. Then

$$Y(s) = N(s)(X(s) + M(s)Q(s))\frac{1}{s}.$$

Since the transfer function is stable, $Y(s)$ only has the simple pole at $s = 0$ and we can apply the final-value theorem to get

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = N(0)(X(0) + M(0)Q(0)) = 1.$$

Similarly, the other asymptotic condition reduces to asking that the transfer function from d to y, which is $N(Y - NQ)$, has a zero for $s = 10j$:

$$N(10j)(Y(10j) - N(10j)Q(10j)) = 0.$$

Asymptotic properties (III)

Thus, we have two algebraic conditions on Q, which after substituting N, M, X and Y become

$$\begin{aligned} Q(0) &= 6, \\ Q(10j) &= -94 + 70j. \end{aligned}$$

A method that will work is to let Q be a polynomial in $\frac{1}{s+1}$ with enough free coefficients to satisfy all the conditions. In this case we have 3 conditions (the second one is complex), so

$$Q(s) = a + b\frac{1}{s+1} + c\frac{1}{(s+1)^2}.$$

Substituting into the equations and solving yields $a = -79$, $b = -723$ and $c = 808$, and then

$$Q = \frac{-79s^2 - 881s + 6}{(s+1)^2}.$$

Finally, the controller is

$$C(s) = \frac{-60s^4 - 598s^3 + 2515s^2 - 1794s + 1}{s(s^2+100)(s+9)}.$$

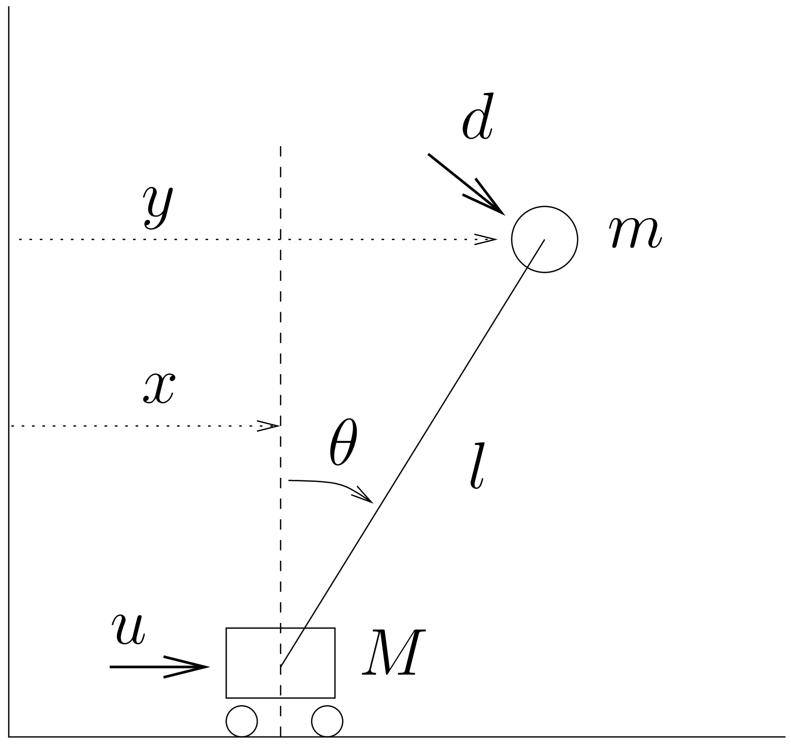
Strong stabilization

- A plant is **strongly stabilizable** if internal stabilization is achieved with a controller that is itself stable.
- The controller from the last example is not stable.
- Stability of the controller itself is something to look for. If the feedback loop breaks, or it is turned off during start-up or shutdown, the overall system has then transfer function CP, which is stable if both the controller and the plant are stable.
- **Theorem 3**. P is strongly stabilizable iff it has an even number of real poles between every pair of real zeros in $\Re(s) \geq 0$ (including $s = \infty$ if P is strictly proper).
- The sufficiency part of the proof is constructive. See Doyle-Francis-Tannenbaum.

Simultaneous stabilization

- Simultaneous stabilization of two plants arises when one tries to stabilize a plant which can change between two transfer functions, P_1 and P_2. It is desirable to have a single stabilizing controller C to avoid switching controllers during operation at essentially unknown times.
- **Theorem 4**. Let $P = P_2 - P_1$. Then P_1 and P_2 are simultaneously stabilizable iff P is strongly stabilizable.

The cart-pendulum (I)



- We take the forces u and d as inputs, x and θ as the state variables, and y as the output.
- There are two equilibrium points corresponding to $\theta = 0$ (pendulum up) and $\theta = \pi$ (pendulum down). The exact equations of motion are nonlinear. We will linearize them around the pendulum up equilibrium point, since this one is unstable and it may be worth trying to stabilize it.

The cart-pendulum (II)

The exact equations of motion are

$$\begin{aligned}(M+m)\ddot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta) &= u,\\ m(\ddot{x}\cos\theta+l\ddot{\theta}-g\sin\theta) &= d,\\ y &= x+l\sin\theta.\end{aligned}$$

Linearization about pendulum up. Drop all the products, put $\sin\theta=\theta$, $\cos\theta=1$. One gets

$$\begin{aligned}(M+m)\ddot{x}+ml\ddot{\theta} &= u,\\ \ddot{x}+l\ddot{\theta}-g\theta &= \frac{1}{m}d,\\ y &= x+l\theta.\end{aligned}$$

The cart-pendulum (III)

Laplace-transforming yields

$$\begin{pmatrix} X \\ \Theta \end{pmatrix} = \frac{1}{D_u(s)} \begin{pmatrix} ls^2 - g & -ls^2 \\ -s^2 & \frac{M+m}{m} s^2 \end{pmatrix} \begin{pmatrix} U \\ D \end{pmatrix}$$

where $D_u(s) = s^2(Mls^2 - (M+m)g)$. Finally

$$Y = X + l\Theta = \frac{1}{D_u(s)} \begin{pmatrix} -g & \frac{M}{m} ls^2 \end{pmatrix} \begin{pmatrix} U \\ D \end{pmatrix}$$

All the transfer functions are unstable, since at least they have the pole at

$$s = +\sqrt{\frac{(M+m)g}{Ml}}.$$

This is in agreement with physical intuition.