

# Lecture 5

## *Design constraints*

This lecture follows Chapter 6 of Doyle-Francis-Tannenbaum. *Complex Variables and Applications*, by Churchill and Brown, can provide the necessary complex function results, which we will briefly review.

# Constraints in the design

- The robust performance condition (RPC) is

$$\| |W_1 S| + |W_2 T| \|_\infty < 1,$$

where  $W_1$ , describing the performance goal, and  $W_2$ , describing the set of plants, are given. One has

$$S = \frac{1}{1 + L}, \quad T = \frac{L}{1 + L},$$

with  $L = PC$ . Find  $C$  so as to satisfy the RPC.

- There are 2 classes of obstacles:
  - algebraic constraints relating  $T$  and  $S$ .
  - analytic constraints on  $|T(j\omega)|$  and  $|S(j\omega)|$  due to the zeros and poles of  $L$ .

# Algebraic constraints

There are three algebraic constraints:

- For every  $s \in \mathbb{C}$ ,  $S(s) + T(s) = 1$ . In particular,  $|S(j\omega)|$  and  $|T(j\omega)|$  cannot both be less than  $1/2$  at the same frequency  $\omega$ .
- Combining the above constraint with the RPC condition, one gets necessarily

$$\min\{|W_1(j\omega)|, |W_2(j\omega)|\} < 1, \quad \forall \omega.$$

- If  $p$  is a pole of  $L$  and  $z$  is a zero of  $L$ , then

$$S(p) = 0, \quad T(p) = 1, \quad S(z) = 1, \quad T(z) = 0.$$

# Complex variable results (I)

- **Maximum modulus theorem.** Let  $\Omega$  be a nonempty, open, connected set in  $\mathbb{C}$  and let  $F$  be analytic in  $\Omega$ . Suppose that  $F$  is not a constant. Then  $|F|$  does not attain a maximum value in  $\Omega$ .

As an application, if  $\Omega$  is the open right half-plane and  $F \in \mathcal{Q}$ , then  $F$  is analytic and of bounded magnitude in the closed set  $\Re(s) \geq 0$ , and thus  $|F|$  has a maximum in  $\Re(s) \geq 0$ . Since it cannot be in  $\Omega$ , it must be on the imaginary axis and hence

$$\|F\|_{\infty} = \sup_{\Re(s) > 0} |F(s)|.$$

# Complex variable results (II)

- **Cauchy's theorem.** Suppose that  $\Omega$  is a bounded open set with connected complement and  $\mathcal{D}$  is a non-self-intersecting closed contour in  $\Omega$ . If  $F$  is analytic in  $\Omega$ , then

$$\oint_{\mathcal{D}} F(s) \, ds = 0.$$

- **Cauchy's integral formula.** Under the same assumptions of Cauchy's theorem, if  $s_0$  is a point in  $\Omega$ , then

$$F(s_0) = \frac{1}{2\pi j} \oint_{\mathcal{D}} \frac{F(s)}{s - s_0} \, ds.$$

# Complex variable results (III)

- **Schwarz's integral formula.** Let  $F$  be analytic and of bounded magnitude in  $\Re(s) \geq 0$ , and let  $s_0 = \sigma_0 + j\omega_0$  with  $\sigma_0 > 0$ . Then

$$F(s_0) = \frac{1}{\pi} \int_{-\infty}^{+\infty} F(j\omega) \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} d\omega.$$

This is also known as Poisson's integral formula for the half-plane. It can be deduced from Cauchy's theorem and Cauchy's integral formula for  $F(s)/(s + \bar{s}_0)$  and  $F(s)/(s - s_0)$  respectively, with  $\mathcal{D}$  the Nyquist contour.

# Analytical constraints (I)

## Bounds on $W_1$ and $W_2$

- If  $z$  is a zero of  $L$  with  $\Re(z) \geq 0$  then, using  $S(z) = 1$ ,

$$|W_1(z)| = |W_1(z)S(z)| \leq \sup_{\Re(s) \geq 0} |W_1(s)S(s)| = \|W_1S\|_\infty,$$

where the maximum modulus theorem has been used.

Hence  $\|W_1S\|_\infty \geq |W_1(z)|$ .

- Similarly, if  $p$  is a pole of  $L$  with  $\Re(p) \geq 0$ ,

$$\|W_2T\|_\infty \geq |W_2(p)|.$$

- It is thus *necessary* that  $|W_1(z)| < 1$  and  $|W_2(p)| < 1$  for performance and for robust stability, respectively.

# Analytical constraints (II)

- A transfer function in  $\mathcal{Q}$  is **all-pass** if its magnitude equals 1 at all points on the imaginary axis.
- Up to a sign, an all-pass function is the product of factors of the form

$$\frac{s - s_0}{s + \bar{s}_0}, \quad \Re(s_0) > 0.$$

- $H(s) = 1, H(s) = -1.$
- $H(s) = \frac{s-1}{s+1}.$
- 

$$H(s) = \frac{s^2 - s + 2}{s^2 + s + 2} = \frac{s - \left(\frac{1}{2} + j\sqrt{\frac{7}{2}}\right)}{s + \left(\frac{1}{2} - j\sqrt{\frac{7}{2}}\right)} \cdot \frac{s - \left(\frac{1}{2} - j\sqrt{\frac{7}{2}}\right)}{s + \left(\frac{1}{2} + j\sqrt{\frac{7}{2}}\right)}.$$

# Analytical constraints (III)

- A transfer function in  $\mathcal{Q}$  is **minimum phase** if it has no zeros in  $\Re(s) > 0$ .
- The name comes from the fact that, given a minimum phase function  $H(s)$ , all the other functions  $G(s)$  with the same magnitude when evaluated on the imaginary line,  $G(s) = H(s)G_{\text{ap}}(s)$ , where  $G_{\text{ap}}$  is all-pass, verify that

$$\arg H(j\omega) \leq \arg G(j\omega) \quad \forall \omega \in \mathbb{R}.$$

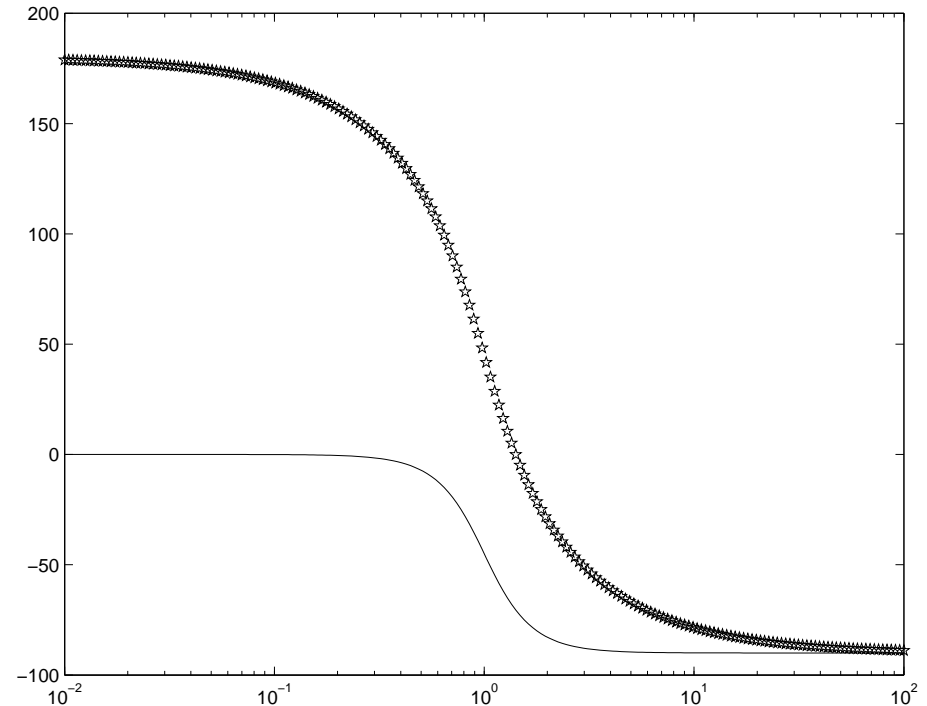
Let

$$H(s) = \frac{s + 1}{s^2 + s + 1}$$

which is minimum-phase, and

$$G(s) = \frac{s - 1}{s^2 + s + 1}$$

which has the same magnitude. The symbol line is the angular part of the Bode plot of  $G$ , while the solid line corresponds to that of  $H$ .



# Analytical constraints (IV)

- **Factorization.** For each function  $G$  in  $\mathcal{Q}$  there exist an all-pass function  $G_{\text{ap}}$  and a minimum-phase function  $G_{\text{mp}}$  such that  $G = G_{\text{ap}}G_{\text{mp}}$ . The factors are unique up to sign.
- $G_{\text{ap}}$  is obtained as the product of all factors of the form

$$\frac{s - s_0}{s + \bar{s}_0}$$

where  $s_0$  ranges over all zeros of  $G$  in  $\Re(s) > 0$ , and then

$$G_{\text{mp}} = \frac{G}{G_{\text{ap}}}.$$

- If

$$G = \frac{s^2 - 3s + 2}{s^2 + s + 1}$$

then

$$G_{\text{ap}} = \frac{s - 2}{s + 2} \cdot \frac{s - 1}{s + 1} = \frac{s^2 - 3s + 2}{s^2 + 3s + 2} \quad \text{and} \quad G_{\text{mp}} = \frac{s^2 + 3s + 2}{s^2 + s + 1}.$$

# Analytical constraints (V)

- For the remainder of this lecture we assume that  $L$  has no poles on the imaginary axis. Remember that  $S$  is not strictly proper (since  $L$  is proper).
- If

$$S = S_{\text{ap}} S_{\text{mp}}$$

then  $S_{\text{mp}}$  has no zeros on the imaginary axis and  $S_{\text{mp}}$  is not strictly proper, since  $S$  is not. Hence  $S_{\text{mp}}^{-1} \in \mathcal{Q}$  because

- $S_{\text{mp}}$  is proper.
- $S_{\text{mp}}$  has no zeros with  $\Re(s) > 0$  (definition of minimum phase).
- $S_{\text{mp}}$  has no zeros with  $\Re(s) = 0$  (assumption).

# Analytical constraints (VI)

- Suppose  $P$  has a zero at  $z$  with  $\Re(z) > 0$ , a pole at  $p$  with  $\Re(p) > 0$ , and no other pole or zero with  $\Re(s) \geq 0$ . Suppose also that  $C$  has no poles or zeros in  $\Re(s) \geq 0$ . Then, since  $p$  and  $z$  are real,

$$S_{\text{ap}}(s) = \frac{s - p}{s + p}, \quad T_{\text{ap}}(s) = \frac{s - z}{s + z}.$$

- From  $T(z) = 0$  and  $S(p) = 0$  we get  $S(z) = 1$  and  $T(p) = 1$ , and hence

$$S_{\text{mp}}(z) = (S_{\text{ap}}(z))^{-1} = \frac{z + p}{z - p}, \quad T_{\text{mp}}(p) = (T_{\text{ap}}(p))^{-1} = \frac{p + z}{p - z}.$$

- Then

$$\|W_1 S\|_{\infty} = \|W_1 S_{\text{mp}}\|_{\infty} \geq |W_1(z) S_{\text{mp}}(z)| = \left| W_1(z) \frac{z + p}{z - p} \right|,$$

and

$$\|W_2 T\|_{\infty} \geq \left| W_2(p) \frac{p + z}{p - z} \right|.$$

- A plant pole and a plant zero near each other in the right half-plane greatly difficult the performance and robustness goals.

# Analytical constraints (VII)

**Example: the cart-pendulum.** Let  $P$  be the transfer function from  $u$  to  $x$ :

$$P(s) = \frac{ls^2 - g}{s^2(Mls^2 - (M + m)g)}.$$

- Setting  $r = m/M$ , we have  $z = \sqrt{\frac{g}{l}}$  and  $p = z\sqrt{1+r}$ .
- Let us focus on the stabilization task. We have

$$\frac{p+z}{p-z} = \frac{\sqrt{1+r}+1}{\sqrt{1+r}-1} > 1$$

$$\text{and } \|W_2T\|_\infty \geq \left| W_2(p) \frac{\sqrt{1+r}+1}{\sqrt{1+r}-1} \right|.$$

- Hence  $r \gg 1$ , *i.e.*  $m \gg M$  seems to facilitate the stabilization task. But this means a large value of  $p$  and hence a large value of  $W_2(p)$  ( $W_2$  is high-pass, typically).
- In contrast, the  $u$  to  $y$  transfer function has no zeros and there is no tradeoff in making  $m/M$  very small.
- Longer sticks are easier to balance on the palm of the hand. It is easier to control an inverted stick by looking at the top than it is by looking at the base.

# Analytical constraints (VIII)

## The waterbed effect.

- Tracking problem with reference signals concentrated in  $[\omega_1, \omega_2]$ . Let

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)|$$

and let  $M_2 = \|S\|_\infty$ .

- The tracking condition  $\|W_1 S\|_\infty < 1$  will be easily attainable if  $M_1 \ll 1$ . On the other hand,

$$\begin{aligned} \frac{1}{M_2} &= \left( \sup_{\omega} \frac{1}{|1 + L(j\omega)|} \right)^{-1} = \inf_{\omega} |1 + L(j\omega)| \\ &= \inf_{\omega} |-1 - L(j\omega)| \\ &= \text{distance from } -1 \text{ to the Nyquist plot of } L, \end{aligned}$$

so, from the robustness point of view, we cannot allow  $M_2$  to be very large. Notice that  $M_2 \geq 1$  because  $|S(j\infty)| = 1$ .

- Question: can we have  $M_1$  small and  $M_2$  not very large?

# Analytical constraints (IX)

## The waterbed effect (cont'd).

- For non-minimum-phase plants, the answer is NO: as  $|S|$  is pushed down in a frequency range, it pops up somewhere else.
- **Theorem (The waterbed theorem).** Suppose that  $P$  has a zero  $z$  with  $\Re(z) > 0$ . Then there exist positive constants  $c_1$  and  $c_2$ , depending only on  $\omega_1$ ,  $\omega_2$  and  $z$ , such that

$$c_1 \log M_1 + c_2 \log M_2 \geq \log |S_{\text{ap}}(z)^{-1}| \geq 0.$$

- If  $z = \sigma_0 + j\omega_0$  and

$$f(\omega) = \frac{1}{\pi} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2},$$

then

$$c_1 = \int_{[-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]} f(\omega) \, d\omega, \quad c_2 = \int_{\mathbb{R} - ([-\omega_2, -\omega_1] \cup [\omega_1, \omega_2])} f(\omega) \, d\omega.$$

# Analytical constraints (X)

## The area formula.

- It turns out that the area bounded by the graph of  $|S(j\omega)|$  in log scale plotted as a function of  $\omega$  in linear scale is bounded from below by the real parts of the unstable poles of  $L$ . Let  $\{p_i\}$  denote the set of poles of  $L$  with  $\Re(p) > 0$ . Remember that *relative degree* equals degree of denominator minus degree of numerator. Then
- **Theorem (The area formula).** Assume that the relative degree of  $L$  is at least 2. Then

$$\int_0^{\infty} \log |S(j\omega)| \, d\omega = \pi \log e \sum \Re(p_i).$$

- This result applies for any system with relative degree of  $L$  at least 2, be it minimum-phase or not.
- Negative area, required for good tracking performance over a given frequency, must be accompanied by positive area.