Lecture 9

Design for stability margin and for robust performance

This lecture is based on chapters 11 and 12 of Doyle-Francis-Tannenbaum.

Optimal robust stability (I)

- Remember the disk multiplicative uncertainty model. We modify it slightly so that the set of plants is now given by

$$\mathcal{P}_\epsilon = \{\tilde{P} = (1 + \Delta W_2)P, \ \ ||\Delta||_\infty \leq \epsilon\},$$

where P is the nominal plant and no unstable pole of P is cancelled when forming $\tilde{P}$. In Lecture 3 we took $\epsilon = 1$.

- Let ϵ_{sup} be the least upper bound on ϵ such that some C stabilizes every plant in $\mathcal{P}_\epsilon$, so ϵ_{sup} is the maximum stability margin for this model of uncertainty.
- The key result in Lecture 3 was that to achieve robust stability for this model

$$||W_2 T||_\infty < \frac{1}{\epsilon}.$$

- Define

$$\gamma_{\text{inf}} = \inf_C ||W_2 T||_\infty,$$

where the infimum is taken over all internally stabilizing controllers. Then

$$\epsilon_{\text{sup}} = \gamma_{\text{inf}}^{-1}.$$

Optimal robust stability (II)

- Computing γ_{inf} reduces to a model-matching problem. Indeed, using the Youla-Kucera parametrization for C, one gets

$$\gamma_{\text{inf}} = \inf_{Q \in \mathcal{Q}} ||W_2 N(X + MQ)||_\infty.$$

- This is similar to the model-matching problem of Lecture 7:

$$\gamma_{\text{opt}} = \min_{Q_{\text{im}} \text{ stable}} ||T_1 - T_2 Q_{\text{im}}||_\infty,$$

  with $T_1 = W_2 N X$, $T_2 = -W_2 N M$.
- So that T_2 has no zeros on the imaginary axis, we will assume that P has neither poles nor zeros on the imaginary axis, and that W_2 has no zeros there.
- The key difference between the two problems is that Q must be stable and proper. However, it can be shown that $\gamma_{\text{inf}} = \gamma_{\text{opt}}$.
- The way to solve the problem is very much like the performance design problem of the previous lecture, using the J_τ functions to take away the improperness of Q_{im}.

UNIVERSITAT POLITÈCNICA DE CATALUNYA UPC

Optimal robust stability (III)

Procedure to solve the robust stability problem. Input: P, W_2.

Step 1. Do a coprime factorization of P over $\mathcal{Q}$.

Step 2. Solve the model-matching problem for $T_1 = W_2 N X$, $T_2 = -W_2 N M$. Let Q_{im} denote its solution, achieving γ_{opt}. Then $\epsilon_{\text{sup}} = 1/\gamma_{\text{opt}}$.

Step 3. Let $\epsilon < \epsilon_{\text{sup}}$. Choose J_τ such that $Q_{\text{im}} J_\tau$ is proper and τ is small enough that

$$||W_2 N(X + M Q_{\text{im}} J_\tau)||_\infty < \frac{1}{\epsilon}.$$

Step 4. Set $Q = Q_{\text{im}} J_\tau$, $C = (X + MQ)/(Y - NQ)$.

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Optimal robust stability (IV)

Example.

- Consider the plant
$$P(s) = \frac{s-1}{(s+1)(s-p)}, \quad 0 < p \neq 1,$$
with an unstable pole at $s = p$ and an unstable zero at $s = 1$.
- Suppose that the uncertainty weight is the high-pass function
$$W_2(s) = \frac{s+0.1}{s+1},$$
so that $|\tilde{P}/P - 1| \sim 0.1\epsilon$ at low frequencies and $|\tilde{P}/P - 1| \sim 1\epsilon$ at high frequencies.
- The coprime factorization of P yields
$$N(s) = \frac{s-1}{(s+1)^2}, \; M(s) = \frac{s-p}{s+1}, \; X(s) = \frac{(p+1)^2}{p-1}, \; Y(s) = \frac{s - \frac{p+3}{p-1}}{s+1}.$$
Notice that X is just a constant.

Optimal robust stability (V)

- Now T_2 is

$$T_2(s) = -W_2(s)N(s)M(s) = -\frac{s+0.1}{s+1}\frac{(s-1)(s-p)}{(s+1)^3}$$

which has 2 unstable zeros. In order to make the model-matching part of the procedure easier, we can decompose N as $N = N_{\rm ap}N_{\rm mp}$ with

$$N_{\rm ap}(s) = \frac{s-1}{s+1}, \quad N_{\rm mp}(s) = \frac{1}{s+1}$$

and keep only $N_{\rm mp}$, since the all-pass part does not contribute to the relevant ∞-norm:

$$||W_2N(X+MQ)||_\infty = ||W_2N_{\rm mp}(X+MQ)||_\infty.$$

The (modified) model-matching problem has data

$$T_1(s) = \frac{(p+1)^2(s+0.1)}{(p-1)(s+1)^2}, \quad T_2(s) = -\frac{(s+0.1)(s-p)}{(s+1)^3}.$$

Optimal robust stability (VI)

- Since now the only unstable zero of T_2 is $s = p$, the solution of the model-matching problem is

$$Q_{\text{im}}(s) = \frac{T_1(s) - T_1(p)}{T_2(s)}$$

and the optimal error is

$$\gamma_{\text{opt}} = |T_1(p)| = \left|\frac{p + 0.1}{p - 1}\right|$$

Thus the maximum unstability margin is

$$\epsilon_{\text{sup}} = \left|\frac{p - 1}{p + 0.1}\right|$$

and goes to zero as p aproaches 1. Hence less and less uncertainty can be tolerated as the unstable pole and zero of the plant approach each other. As we already know from Lecture 5, this is a general fact.

- To proceed, let's take $p = 0.5$, for which $\epsilon_{\text{sup}} = 0.8333$ and

$$Q_{\text{im}}(s) = -1.2\frac{(s + 1)(s - 1.25)}{s + 0.1}.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Optimal robust stability (VII)

- We arbitrarily set $\epsilon = 0.8$ and, since the relative degree of Q_{im} is -1,

$$J_\tau(s) = \frac{1}{\tau s + 1}.$$

- We try several values of τ and for $\tau = 0.01$ we get

$$||W_2 N(X + MQ_{\text{im}} J_{0.01}||_\infty = 1.2396 < 1.25 = \frac{1}{\epsilon}.$$

- Finally

$$Q(s) = -1.2\frac{(s+1)(s-1.25)}{(s+0.1)(0.01s+1)}$$

and

$$C(s) = -\frac{(s+1)(124.5s^2 + 240.45s + 120)}{s^3 + 227.1s^2 + 440.7s + 220}.$$

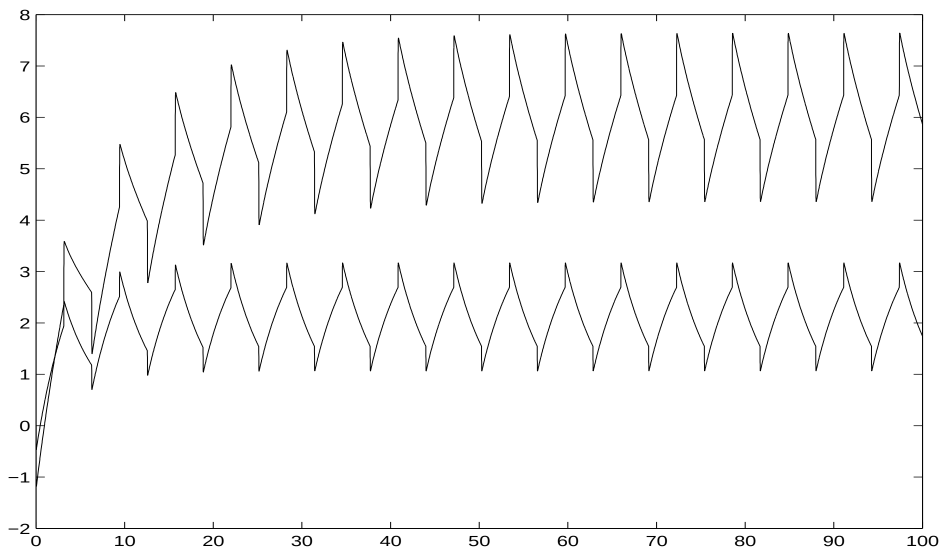
UNIVERSITAT POLITÈCNICA DE CATALUNYA
UPC

Optimal robust stability (VIII)

- We can try the above controller for some plants in $\mathcal{P}_\epsilon$. Let's consider plants of the form

$$\tilde{P} = k\frac{s-a}{(s+1)(s-0.5)}.$$

- It can be shown that if k and a satisfy $(ka-1)^2 + k^2(\omega-1)^2 < 0.64(\omega^2+1)$ for all ω, then $\tilde{P}$ belongs to $\mathcal{P}_{0.8}$. This essentially requires $|k| < 0.8$.
- Let us take $k = 0.6$ and $a = 2$. The response to a square periodic pulse of the nominal plant (upper) and of the perturbed one (lower) is displayed below, and both are clearly bounded.



Gain margin optimization (I)

- We now turn to a different model of plant uncertainty, the gain uncertainty model, which we have already encountered and which is commonly used in elemental control theory.
- To be precise, the set of plants is given by

$$\mathcal{P} = \{\tilde{P} = kP,\ 1 \leq k \leq k_1\}$$

  for a given k_1. This corresponds to $W_2(s) = k_1 - 1$ in the disk multiplicative model, but more precise results can be given for this particular case.
- Let k_{sup} be the supremum value of the k_1 such that a controller exists achieving internal instability for the set of plants. We will present a formula for k_{sup}, assuming that P has neither poles nor zeros on the imaginary axis.
- Define the infimum norm of the (unweighted) complementary sensitivity function

$$\gamma_{\text{inf}} = \inf_C \|T\|_\infty .$$

- **Lemma.** One has that $\gamma_{\text{inf}} = 0$ if P is stable, $\gamma_{\text{inf}} = 1$ if P is unstable but minimum phase, and $\gamma_{\text{inf}} > 1$ if P is unstable but non-minimum phase.

UNIVERSITAT POLITÈCNICA DE CATALUNYA
UPC

Gain margin optimization (II)

Theorem 1. If P is stable or minimum phase, then $k_{\text{sup}} = \infty$. Otherwise

$$k_{\text{sup}} = \left(\frac{\gamma_{\text{inf}} + 1}{\gamma_{\text{inf}} - 1}\right)^2 .$$

The fact that $k_{\text{sup}} = \infty$ or ortherwise does not mean that a single controller will internaly stabilize all the plants with $k < k_{\text{sup}}$; however, given $k_1 < k_{\text{sup}}$, a controller can be computed which works for all the $k \leq k_1$.

The proof of the theorem is fairly involved, and uses some conformal mapping theory. From the proof, a method can be extracted to compute the controller. For P unstable and non-minimum phase (and with no imaginary axis poles or zeros) it goes like this:

Step 1. Do a coprime factorization of P.

Step 2. Solve the model-matching problem for $T_1 = NX$, $T_2 = -NM$. Let Q_{im} denote its solution and let γ_{opt} denote the minimum model-matching error. Then

$$k_{\text{sup}} = \left(\frac{\gamma_{\text{opt}} + 1}{\gamma_{\text{opt}} - 1}\right)^2 .$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Gain margin optimization (III)

Step 3. Let k_1 be arbitrary with $1 < k_1 < k_{\text{sup}}$. Set $J_\tau(s) = \frac{1}{(\tau s+1)^n}$ with n large enough so that $Q_{\text{im}} J_\tau$ is proper and τ small enough so that

$$||N(X + MQ_{\text{im}} J_\tau)||_\infty < \frac{\sqrt{k_1}+1}{\sqrt{k_1}-1}.$$

Step 4. Set

$$\begin{aligned}
K &= N(X + MQ_{\text{im}} J_\tau), \\
G &= \frac{1-\sqrt{k_1}}{1+\sqrt{k_1}} K, \\
T &= \frac{1}{k_1 - 1}\left(\left(\frac{1-G}{1+G}\right)^2 - 1\right), \\
Q &= \frac{T - NX}{NM}.
\end{aligned}$$

Step 5. Get C using Youla-Kucera and the above Q.

UNIVERSITAT POLITÈCNICA DE CATALUNYA UPC

Gain margin optimization (IV)

As an example, let us return to the plant

$$P(s) = \frac{s-1}{(s+1)(s-p)}, \quad 0 < p \neq 1$$

and study now the gain margin problem, $\tilde{P} = kP$.
The coprime decomposition yields

$$N(s) = \frac{s-1}{(s+1)^2},\ M(s) = \frac{s-p}{s+1},\ X(s) = \frac{(p+1)^2}{p-1},\ Y(s) = \frac{s-(p+3)/(p-1)}{s+1}.$$

Let us factor N as $N = N_{\text{ap}} N_{\text{mp}}$ with

$$N_{\text{ap}}(s) = \frac{s-1}{s+1}, \quad N_{\text{mp}}(s) = \frac{1}{s+1}$$

and consider the equivalent model-matching problem with

$$T_1 = N_{\text{mp}} X = \frac{(p+1)^2}{(p-1)(s+1)}, \quad T_2 = -N_{\text{mp}} M = -\frac{s-p}{(s+1)^2}.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA UPC

Gain margin optimization (V)

We get

$$\gamma_{\text{opt}} = |T_1(p)| = \left|\frac{p+1}{p-1}\right|$$

and

$$k_{\text{sup}} = \left(\frac{p+1+|p-1|}{p+1-|p-1|}\right)^2 = \begin{cases} p^2 & p \geq 1 \\ 1/p^2 & p < 1 \end{cases}$$

which, as in the case of ϵ_{sup} for the disk multiplicative model, as a minimum at $p = 1$.
Let us take arbitrarily $p = 2$, for which $k_{\text{sup}} = 4$. The model-matching problem yields, since T_2 has a single unstable zero,

$$Q_{\text{im}} = \frac{T_1 - T_1(p)}{T_2} = 3(s+1).$$

Again arbitrarily, let us set $k_1 = 3.5$. With $J_\tau(s) = 1/(\tau s + 1)$, the value $\tau = 0.01$ yields

$$||N(X + MQ_{\text{im}}J_{0.01}||_\infty = 3.0827 < \frac{\sqrt{k_1}+1}{\sqrt{k_1}-1} = 3.2967.$$

Finally $K(s)$, $G(s)$, $T(s)$ and $Q(s)$ could be computed, and then $C(s)$ from Youla-Kucera.

UNIVERSITAT POLITÈCNICA DE CATALUNYA
UPC

Phase margin optimization

- Now the set of plants to be stabilized is

$$\mathcal{P} = \left\{ \tilde{P} = e^{-j\theta} P, -\theta_1 \leq \theta \leq \theta_1 \right\},$$

where P is the nominal plant and $\theta_1 \in (0, \pi]$.

- Let θ_{sup} denote the supremum θ_1 such that a stabilizing controller for the set does exist. As in the preceeding section, let $\gamma_{\text{inf}} = \inf_C ||T||_\infty$.
- Under the assumption that P has neither poles nor zeros on the imaginary axis, we have the

Theorem 2. If P is stable or minimum phase, then $\theta_{\text{sup}} = \pi$. Otherwise

$$\theta_{\text{sup}} = 2 \arcsin \frac{1}{\gamma_{\text{inf}}}.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA

UPC

The modified problem (I)

- Remember that the robust performance problem is to design a (proper) controller so that the feedback system for the nominal plant is internally stable and the inequality (RPT)

$$\| \, |W_1 S| + |W_2 T| \, \|_\infty < 1$$

holds.

- As stated, the problem has not been solved in general, so we look for a nearby problem that is solvable.
- Fix a frequency and let $x = |W_1 S|$, $y = |W_2 T|$. Then

$$x^2 + y^2 < \frac{1}{2} \Rightarrow x + y < 1.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA UPC

The modified problem (II)

- Thus, a sufficient condition for the RPT to hold is the **modified robust performance test** (MRPT)

$$|| \, |W_1 S|^2 + |W_2 T|^2 \, ||_\infty < \frac{1}{2}.$$

- Notice that it is entirely possible for the MRPT to have no solution and yet the RPT be solvable.
- We will try to solve the MTPT under the following simplifying assumptions
  - P is strictly proper and has neither poles nor zeros on the imaginary axis.
  - W_1 and W_2 are stable and proper, and have no common zeros on the imaginary axis.

Spectral factorization (I)

- For a rational function $F(s)$ with real coefficients, let $\overline{F}(s) = F(-s)$. This is the complex conjugate value when $s = j\omega$. Thus

$$\overline{F}(j\omega) = F(-j\omega) = \overline{F(j\omega)}.$$

- We saw that, if $F \in \mathcal{Q}$, it has a factorization of the form $F = F_{\text{ap}} F_{\text{mp}}$. The all-pass factor has the property

$$\overline{F}_{\text{ap}}(s) F_{\text{ap}}(s) = 1.$$

- When $\overline{F} = F$ and F has no zeros or poles on the imaginary axis there is a related factorization, called **spectral factorization**.

UNIVERSITAT POLITÈCNICA DE CATALUNYA
UPC

Spectral factorization (II)

- If $\overline{F} = F$ and no poles or zeros on the imaginary axis, we can write

$$F(s) = cF_1(s), \quad F_1(s) = \frac{\prod(z_i - s)(z_i + s)}{\prod(p_i - s)(p_i + s)},$$

  where $\{z_i\}$ and $\{p_i\}$ are the right half-plane zeros and poles. Note that $F_1(0) > 0$, since the F is real-rational and complex zeros or poles must appear in conjugate pairs.

- From F_1 form a function G by selecting the poles and zeros in $\Re(s) < 0$:

$$G(s) = \frac{\prod(z_i + s)}{\prod(p_i + s)}.$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA
UPC

Spectral factorization (III)

- With this G we have

$$F(s) = \overline{G}(s) c G(s), \quad \text{with } G, G^{-1} \text{ stable.}$$

- Finally, if $c > 0$, we define an **spectral factor of** F, F_{sf}, as

$$F_{\text{sf}}(s) = \sqrt{c} \frac{\prod (z_i + s)}{\prod (p_i + s)}.$$

  Note that $c > 0$ iff $F(0) > 0$.

- Hence we have a **spectral factorization** (it is not unique)

$$F = \overline{F}_{\text{sf}} F_{\text{sf}}, \quad \text{with } F_{\text{sf}} \text{ and } F_{\text{sf}}^{-1} \text{ stable.}$$

UNIVERSITAT POLITÈCNICA DE CATALUNYA

UPC

Solution of the MRPT (I)

- The modified RPT can be transformed into a model-matching problem using an spectral factorization.
- In terms of the Youla-Kucera parametrization, the MRPT is

$$\| \, |W_1M(Y - NQ)|^2 + |W_2N(X + MQ)|^2 \, \|_\infty < \frac{1}{2}. \quad (1)$$

- Setting $R_1 = W_1MY$, $R_2 = W_1MN$, $S_1 = W_2NX$, $S_2 = -W_2MN$, (1) becomes

$$\| \, |R_1 - R_2Q|^2 + |S_1 - S_2Q|^2 \, \|_\infty < \frac{1}{2}. \quad (2)$$

Solution of the MRPT (II)

- The first key step is to find $U_1, U_2 \in \mathcal{Q}$, and U_3 real rational and satisfying $\overline{U}_3 = U_3$ such that (2) becomes

$$|| \, |U_1 - U_2 Q|^2 + U_3 \, ||_\infty < \frac{1}{2}. \tag{3}$$

- The second key step is to introduce U_4, a spectral factor of $\frac{1}{2} - U_3$. Then (2) can be written as

$$||U_4^{-1} U_1 - U_4^{-1} U_2 Q||_\infty < 1. \tag{4}$$

This is already an standard model-matching problem, and the, by now, well known machinery can be started.

UNIVERSITAT POLITÈCNICA DE CATALUNYA

UPC

Solution of the MRPT (III)

The whole procedure can be partitioned as follows. First goes a routine to compute U_1 and U_2.

Procedure A. Given R_1, R_2, S_1, S_2,

Step A1. Set $F = \overline{R}_2 R_2 + \overline{S}_2 S_2$.

Step A2. Compute a spectral factor F_{sf} of F.

Step A3. Choose an all-pass funtion V such that

$$\frac{\overline{R}_2 R_1 + \overline{S}_2 S_1}{\overline{F_{\text{sf}}}} V \in \mathcal{Q}.$$

Step A4. Set

$$U_1 = \frac{\overline{R}_2 R_1 + \overline{S}_2 S_1}{\overline{F_{\text{sf}}}} V, \quad U_2 = F_{\text{sf}} V.$$

Solution of the MRPT (IV)

The main procedure is as follows.

Procedure. Given P, W_1, W_2,

Step 1. Compute

$$U_3 = \frac{\overline{W_1}W_1\overline{W_2}W_2}{\overline{W_1}W_1 + \overline{W_2}W_2}.$$

Check if $||U_3||_\infty < 1/2$. If not, the problem is not solvable; exit.

Step 2. Do a coprime factorization of P. Get N, M, X and Y.

Step 3. Set $R_1 = W_1 MY$, $R_2 = W_1 MN$, $S_1 = W_2 NX$, $S_2 = -W_2 MN$.

Step 4. Apply Procedure A to get U_1 and U_2.

Solution of the MRPT (V)

Step 5. Compute a spectral factor, U_4, of $\frac{1}{2} - U_3$ (this is guaranteed to exist by Step 1).

Step 6. Set $T_1 = U_4^{-1}U_1$, $T_2 = U_4^{-1}U_2$.

Step 7. Compute γ_{opt} for the model-matching problem of Step 6. If $\gamma_{\text{opt}} < 1$ continue; otherwise the MRPT is not solvable; exit.

Step 8. Compute Q, the solution to the above model-matching problem. If Q is not proper, roll it off at high frequency while maintaining $||T_1 - T_2Q||_\infty < 1$.

Step 9. Get $C = (X + MQ)/(Y - NQ)$.