The Spectra of Wrapped Butterfly Digraphs

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The knowledge of the spectrum of a (di)graph is relevant for estimating some of its structural properties, which provide information on the topological and communication properties of the corresponding networks. Among these properties, we have, for instance, edge-expansion and node-expansion, bisection width, diameter, maximum cut, connectivity, and partitions. In this paper, we determine the complete spectra (eigenvalues and multiplicities) of wrapped butterfly digraphs. © 2003 Wiley Periodicals, Inc.

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1. INTRODUCTION

Butterfly networks have been extensively studied in the literature because of their many applications in computer architectures. In the undirected case, aspects that have been considered include, for example, the study of the cycle structure and Hamiltonicity [1, 20, 26] and the development of communication and routing algorithms [10, 12, 21–23]. Some of these results have been recently extended also to the directed case [2, 3].

The knowledge of the spectrum of a graph is important as spectral results are relevant for the estimation of connectivity and expansion parameters of graphs, which are, in general, very hard to obtain by other methods. In particular, the spectrum of the adjacency matrix of a graph gives direct bounds on the diameter and the isoperimetric number [8, 24], among other properties. The Laplacian spectrum, in particular, the second-smallest eigenvalue, contains information on the connectivity, diameter, expanding properties, maximum cut, independence number, genus, bisection width, etc. [4, 6, 9, 25]. Efficient graph partition algorithms have been constructed based on the eigenvalues and eigenvectors [17, 18]. The spectrum of a graph has also been considered in load-balancing algorithms [4, 14].

The spectra of undirected unwrapped butterfly graphs was determined recently in [15] by considering the hierarchical structure of the graphs. Here, the study of walks on a wrapped butterfly digraph will allow us to completely characterize its spectrum. The knowledge of this spectrum will facilitate a new approach for a better characterization of this important category of networks.

2. NOTATION

The wrapped butterfly digraph $B_\Delta(n)$, of degree $\Delta$ and dimension $n$, has vertices labeled by ordered pairs $(l, x)$, where $l \in \mathbb{Z}_n$ (l is called the level) and $x \in \mathbb{Z}_\Delta^n$; vertex $(l, x_0 x_1 \cdots x_{n-1})$ is adjacent to the vertices $(l + 1, x_0 \cdots$
In this section, we obtain the distance matrices of a wrapped butterfly digraph as polynomials in the adjacency matrix. These polynomials will be used in the next section to determine the eigenvalues of the digraph and their multiplicities.

Let us first notice that in \( B_\Delta (n) \) every edge joins vertices in consecutive levels and two adjacent vertices \((l, x)\) and \((l', y)\) are such that \(x\) and \(y\) differ only in one digit. Notice also that vertex \((l, x)\) has all its adjacent vertices at level \(l + 1 \pmod{n}\) and two different vertices at the same level, \((l, x)\) and \((l', y)\), are at distance \(n\). The next lemma provides a way to compute the length of a walk between any two vertices:

**Lemma 1.** The length of any walk from vertex \((l, x)\) to vertex \((l', y)\) in \( B_\Delta (n) \) equals \( l' - l \) modulo \( n \).

**Proof.** Consider that \(0 \leq l \leq l' \leq n\). If \(x_i = y_i\), for each \(i\), \(0 \leq i < l\) or \(l' \leq i < n\), the length of the shortest path is \(l' - l\):

\[
(l, x_0 \cdots x_{l-1}x_lx_{l+1} \cdots x_{n-1}) \rightarrow (l + 1, x_0 \cdots x_{l-1}y_0x_{l+1} \cdots x_{n-1}) \rightarrow \cdots \rightarrow (l' - 1, x_0 \cdots x_{l-1}y_{l'-2} \cdots x_{n-1}) \rightarrow (l', x_0 \cdots x_{l'-1}y_{l'-1}x_{l'-2} \cdots x_{n-1});
\]

otherwise, \(n\) more steps are needed to reach vertex \((l', y)\).

We can use a similar argument when \(0 \leq l' < l \leq n\). We have

\[
\begin{align*}
\text{if } x_i = y_i, \quad &0 \leq i < l, \quad l' \leq i < n &\quad \text{if } x_i = y_i, \quad &l' \leq i < l \\
\text{otherwise,} & &\text{otherwise.}
\end{align*}
\]

The next results follow from the definition of butterfly digraphs and from Lemma 1.

**Lemma 2.** The wrapped butterfly digraph \( B_\Delta (n) \) has girth \( n \).

**Lemma 3.** Given any pair of different vertices, \((l, x)\) and \((l', y)\), in \( B_\Delta (n) \) with \(d(l, x), (l', y)\) = \(k \leq n\), the shortest path from \((l, x)\) to \((l', y)\) is unique.

Any vertex \((l, x)\) in \( B_\Delta (n) \) has \(\Delta\) neighbors at level \(l + 1\) and each of them has again \(\Delta\) adjacent vertices at level \(l + 2\) and so on. Thus, for \(0 \leq k < n\), the number of vertices at distance \(k\) from \((l, x)\) is the maximum possible for a \(\Delta\)-regular digraph with girth \(g = n\), namely, \(\Delta^k\). When \(k = n\), all the vertices at distance \(n\) from \((l, x)\) are at the same level \(l\) and vertex \((l, x)\) is in a unique cycle of length \(n\); therefore, there are \(\Delta^n - 1\) vertices at distance \(n\) from \((l, x)\).

The following lemma counts the number of walks of length \(k\), \(n + 1 \leq k \leq 2n - 1\), from vertex \((l, x)\) to any vertex \((l', x)\).

**Lemma 4.** Given \( B_\Delta (n) \), the number of walks of length \(k\), \(n + 1 \leq k \leq 2n - 1\), from vertex \((l, x)\) to any vertex \((l', x)\) is \(\Delta^{k-n}\).

**Proof.** For \(k = n + 1\), the walks of length \(n + 1\) from vertex \((l, x)\) to \((l', x)\) must be

\[
(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1});
\]

where \(\alpha_{n+1} = x\) and \(\alpha_i\) is the least non-negative integer such that

\[
\alpha_i \cdot \Delta + (\alpha_i + 1) \equiv \alpha_{i+1} \pmod{n}, \quad 0 \leq i < n.
\]
Consequently, the distance matrices $A_n$ from $(l, x_0)$ to $(l', x_0)$ in the adjacency matrix $A$ of the butterfly digraph $B_4(n)$ is the $n$-th root of the unit $\Delta$. We rewrite the previous expressions in terms of such polynomials, named the distance polynomials of the digraph $B_4(n)$:

\[
p_k(x) = \begin{cases} 
1 & \text{if } 0 \leq k < n \\
\Delta^{-n} x^k - x^{k-n} & \text{if } n \leq k \leq 2n - 1.
\end{cases}
\]

A weakly distance regular digraph is a connected digraph $\Gamma$ of diameter $D$ such that the number of walks of a given length $0 \leq k \leq D$ between any two vertices in $\Gamma$ only depends on their distance $[11]$. One characterization for a weakly distance regular digraph $\Gamma$ is that each matrix power $A^k$, $0 \leq k \leq D$, can be expressed as a linear combination of the distance matrices of $\Gamma$. Thus, $B_4(n)$ is a weakly distance regular digraph.

4. THE SPECTRUM OF $B_4(n)$

From the definition of the distance-$k$ matrices, it follows that $\sum_{k=0}^{2n-1} A_k = J$, where $J$ denotes the all-1 matrix. On the other hand, a digraph $\Gamma$ is (strongly) connected and regular if and only if there is a polynomial $p(x) \in \mathbb{Q}[x]$ such that $p(A) = J$ (see [19] or [7], Th. 5.3.1). The polynomial $H_\Gamma(x)$ of least degree satisfying this property is called the Hoffman polynomial of $\Gamma$ and it is given by

\[
H_\Gamma(x) = \frac{N(x)}{S(\Delta)},
\]

where $m_\Gamma(x) = (x - \Delta)S(x) \in \mathbb{Z}[x]$ is the minimum polynomial of $\Gamma$, $\Delta$, the degree of $\Gamma$; and $N$, its order. It is known that the degree of $H_\Gamma(x)$ is at least $D$, where $D$ is the diameter of $\Gamma$.

Theorem 2. The adjacency spectrum of the wrapped butterfly digraph $B_4(n)$ is

\[
\lambda_j = \Delta \omega^j \quad (0 \leq j \leq n - 1) \quad \text{and} \quad \lambda_n = 0,
\]

where $\omega := e^{i2\pi/n}$ is any primitive $n$-th root of unity, with respective multiplicities

\[
m(\lambda_j) = 1 \quad (0 \leq j \leq n - 1) \quad \text{and} \quad m(\lambda_n) = N - n,
\]

where $N = n\Delta^n$.

Proof. Since $\sum_{k=0}^{2n-1} p_k(A) = \sum_{k=0}^{2n-1} A_k = J$ and the degree of $S(x) = \sum_{k=0}^{2n-1} p_k(x)$ is equal to the diameter of $B_4(n)$, we have that $S(x)$ is the Hoffman polynomial of $B_4(n)$. Therefore, the distinct eigenvalues of $B_4(n)$, apart from $\lambda_0 = \Delta$, are the zeros of

\[
S(x) = x^n \left(1 + \frac{x}{\Delta} + \left(\frac{x}{\Delta}\right)^2 + \ldots + \left(\frac{x}{\Delta}\right)^{n-1}\right),
\]

that is, $\lambda_j = \Delta \omega^j$, $1 \leq j \leq n - 1$, and $\lambda_n = 0$.

So, there are $n + 1$ distinct eigenvalues. Their respective
multiplicities can be determined by solving the system of linear equations

$$\text{tr } A' = \sum_{j=0}^{n} m(\lambda_j) \lambda_j' = N a_{uu}^{(j)} \quad (0 \leq j \leq n),$$

where $a_{uu}^{(j)}$ represents the number of walks $u \to u$ of length $j$, which is $a_{uu}^{(j)} = 1$, if $j = 0$, $n$, and $a_{uu}^{(j)} = 0$ otherwise. The coefficient matrix is a Vandermonde matrix formed from the distinct points $\lambda_0, \lambda_1, \ldots, \lambda_n$:

$$
\begin{pmatrix}
0 & 1 & 1 & \cdots & 1 \\
\Delta & \Delta \omega & \cdots & \Delta \omega^{n-1} & 0 \\
\Delta^2 & \Delta^2 \omega^2 & \cdots & \Delta^2 \omega^{n-2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta^n & \Delta^n \omega^n & \cdots & \Delta^n \omega^{n(n-1)} & 0 \\
\end{pmatrix}
\begin{pmatrix}
m(\lambda_0) \\
m(\lambda_1) \\
m(\lambda_2) \\
\vdots \\
m(\lambda_n) \\
\end{pmatrix}

From the first and last equations, and taking into account that $\omega^2 = 1$, we obtain $m(\lambda_0) = N - n$. As a consequence, since $\sum_{j=1}^{n-1} m(\lambda_j) = N$ and $m(\lambda_j) \geq 1$, we deduce that $m(\lambda_j) = 1$, $0 \leq j \leq n - 1$.

The Laplacian spectrum of a $\Delta$-regular digraph consists of all values $\lambda_\Delta = \Delta - \lambda_A$, where $\lambda_A$ is an eigenvalue of the adjacency matrix, and $\Delta$, the degree of the digraph. Therefore, the next result follows:

**Theorem 3.** The Laplacian spectrum of the wrapped butterfly digraph $B_\Delta(n)$ is

$$\lambda_j = \Delta - \Delta \omega^j \quad (0 \leq j \leq n - 1) \quad \text{and} \quad \lambda_n = \Delta,$$

where $\omega := e^{2\pi i/n}$ is any primitive $n$-th root of unity, with respective multiplicities

$$m(\lambda_j) = 1 \quad (0 \leq j \leq n - 1) \quad \text{and} \quad m(\lambda_n) = N - n.$$

**Remark**

Dr. Charles Delorme (LRI, Université Paris-Sud) noticed that our methods enable the computation of spectra for a larger class of digraphs, which we have identified as digraph conjunctions. In general, the conjunction $Y := \Gamma_1 \otimes \Gamma_2$ of digraphs $\Gamma_1$ and $\Gamma_2$, with adjacency matrices $A_1$ and $A_2$, is a digraph which has as adjacency matrix the Kronecker product $A_1 \otimes A_2$. Hence, if $\Gamma_1$ has an eigenvalue $\lambda_1$ with multiplicity $m(\lambda_1)$ and $\Gamma_2$ has the eigenvalue $\lambda_2$ with multiplicity $m(\lambda_2)$, then $Y$ will have the eigenvalue $\lambda_1 \lambda_2$ with multiplicity $m(\lambda_1)m(\lambda_2)$. Therefore, the wrapped butterfly digraph $B_\Delta(n)$ is the conjunction of the (directed) cycle $C_n$ and the de Bruijn digraph $B(\Delta, n)$. The eigenvalues of a directed cycle with $n$ vertices are the $n$ primitive roots of unity and the eigenvalues of the de Bruijn digraph are $\Delta$ and 0 with multiplicity $n - 1$. The result from Theorem 2 then follows. Instead of these digraphs, one could also consider other families of digraphs with well-known spectra, such as a cycle of any size, an exactly $k$-reachable digraph (with adjacency matrix $A$ satisfying $A^k = J$), Kautz digraphs, Moore digraphs, etc. However, such conjunctions do not produce, in general, weakly distance-regular digraphs and, thus, their path-structural analysis seems to deserve less attention.

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REFERENCES


