EXPLICIT GEOMETRY ON A FAMILY OF CURVES OF GENUS 3

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Abstract
We present an explicit geometrical study of the curves

\[ C_a : Y^4 = X^4 - (a^2 + a^{-2})X^2 + 1, \quad a \in \mathbb{R}, a \neq 0, \pm 1. \]

These are non-singular curves of genus 3, defined over \( \mathbb{Q}(a) \). Exploiting their symmetries, we are able to determine most of their geometric invariants, such as their bitangent lines and their period lattice. We give an explicit description of the bijection induced by the Abel-Jacobi map between their bitangent lines and odd 2-torsion points on their jacobian. Finally, we construct three elliptic quotients of these curves, which provide a splitting of their jacobians. In the case of curve \( C_{1 \pm \sqrt{2}} \), which is isomorphic to the Fermat curve of degree 4, our computations yield a finer splitting of its jacobian than the classical one.

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1 Introduction

Curves of genus 3 have the agreeable property that their canonical embedding gives a plane model of them. This characteristic provides a way of finding good references to reflect their symmetries: given a curve of genus 3 with some automorphisms, we may find a basis for the space of holomorphic differentials such that the automorphisms of the curve have easy matrices with respect to it. If we find such a basis, we can make a large number of computations for the curve.

We apply this idea to the projective curves \( C_a \) given by

\[
C_a : Y^4 = (X^2 - a^2 Z^2)(X^2 - a^{-2} Z^2), \quad a \in \mathbb{R}, a \neq 0, \pm 1. \tag{1}
\]

These are non-singular curves of genus 3, defined over \( \mathbb{Q}(a) \). We note that the parameters \( \pm a, \pm a^\pm 1 \) give rise to the same curve. Let \( n = \left( \frac{a+a^{-1}}{2} \right)^2, \kappa = a^2 + 1, \lambda = \frac{1-a^4}{4a^2} \). The curve \( C_a \) is isomorphic over \( \mathbb{Q}(a, \sqrt{\lambda}) \) to the curve

\[
Y^4 = XYZ(X-Z)(X-nZ), \tag{2}
\]

an isomorphism being given by \((x, y, z) \mapsto (\kappa \frac{x}{a} - z, y\sqrt{\lambda}, ax - z)\). This family of curves has been studied under the parametrization (2) in [K-K 79]. Parametrization (1) provides a better reflection of their symmetries and allows a deeper study of the curves. We can readily determine their bitangent lines and their automorphism group. But of course we can go further and, for instance, determine a basis for the singular homology of the curve. Then, by means of elliptic calculi, we are able to calculate the period lattice of the curves. Having done so, we give an explicit description of the bijection between their bitangent lines and odd 2-torsion points on their jacobian induced by the Abel-Jacobi map.

The curves \( C_a \) are a special case of the curves constructed by Cassels in [Ca 85] from a quadratic form. It is asserted in loc.cit. that the jacobians of these curves split completely, i.e., they are isogenous to products of elliptic curves. We provide the explicit isogeny and its kernel for our curves \( C_a \). For the Fermat curve of degree 4, which is isomorphic to the curve \( C_{\pm \sqrt{2}} \), our computations yield a finer splitting of its jacobian than the classical one.

I wish to express my gratitude to P. Bayer for her encouragement during the preparation of this work.
2 Bitangent lines and automorphism group

We begin our study of the curves $C_a$ by giving their bitangent lines. As is well known, a curve of genus three has 28 bitangent lines.

**Proposition 2.1.** The twenty-eight bitangent lines to the curve $C_a$ are given by:

\[ t_{\pm a^{\pm 1}} : X = \pm a^{\pm 1}Z, \]
\[ L_r : Y = \left(\frac{1+i}{\sqrt{2}}\right)^r \sqrt{\frac{a^2-a-2}{2}} X, \quad r = 1, 3, 5, 7, \]
\[ L_r' : Y = \left(\frac{1+i}{\sqrt{2}}\right)^r \sqrt{\frac{a^2-a-2}{2}} Z, \quad r = 1, 3, 5, 7, \]
\[ \ell_{rs} : Y = \left(\sqrt{\frac{a^{1-a-1}}{a-a}}\right)^{(-1)^r} i^s (X + ivZ), \quad r, s = 0, 1, 2, 3. \]

**Proposition 2.2.** For $a \neq \pm 1 \pm \sqrt{2}$ the only quadritangent lines to the curve $C_a$ are the lines $t_{\pm a}, t_{\pm 1/a}$. The curve $C_{\pm 1 \pm \sqrt{2}}$ has eight quadritangent lines more, the lines $\ell_{1s}, \ell_{3s}$, $s = 0, 1, 2, 3$.

The twenty-eight bitangent lines to the curves $C_a$ are distributed in seven sets of four intersecting lines each. The automorphisms of $C_a$ are restrictions of projectivities of $\mathbb{P}^2(\mathbb{C})$. Their dualizations act on the set of intersection points of the bitangent lines. This provides an easy way of determining the automorphism group of $C_a$:

**Proposition 2.3.** Let $a \neq 0, \pm 1, \pm 1 \pm \sqrt{2}$. The automorphisms of $C_a$ are the restrictions of the following projectivities of $\mathbb{P}^2(\mathbb{C})$:

\[ \varphi_{0k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi_{1k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^k & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad k = 0, 1, 2, 3. \]

\[ \varphi_{2k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \varphi_{3k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i^k & 0 \\ -1 & 0 & 0 \end{pmatrix}. \]

The group $\text{Aut}(C_a)$ is isomorphic to a semidirect product $\mathbb{Z}/4\mathbb{Z} \ltimes V_4$. The automorphisms

\[ \alpha(x, y, z) = (x, iy, z), \]
\[ \beta(x, y, z) = (-x, y, z), \]
\[ \gamma(x, y, z) = (z, y, x). \]

form a system of generators for $\text{Aut}(C_a)$. 
The particular behaviour of the curve $C_{\pm 1 \pm \sqrt{2}}$ is not surprising, because, in fact, it is isomorphic to the Fermat curve of degree four:

**Proposition 2.4.** The curve $C_{\pm 1 \pm \sqrt{2}}$ is isomorphic to the Fermat curve of fourth degree, $F_4 = \{Y^4 = X^4 + Z^4\}$. The group $\text{Aut}(F_4)$ has order 96, and is generated by the restrictions of the following projectivities of $\mathbb{P}^2(\mathbb{C})$:

$$A = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1+i}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1+i}{\sqrt{2}} \\ 0 & 0 & 1 \end{pmatrix}.$$  

The group $\text{Aut}(F_4)$ is isomorphic to a semidirect product $(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \rtimes S_3$.

An isomorphism between $C_{\pm 1 \pm \sqrt{2}}$ and $F_4$ is given by

$$F_4 \longrightarrow C_{\pm 1 \pm \sqrt{2}} \quad (x, y, z) \longrightarrow (x + z, \sqrt[4]{8y}, i(x - z)).$$

**3 Homology**

We will suppose from now on that $a \in \mathbb{R}$, and, without loss of generality, that $a > 1$. Let us view the curve $C_a$ as a ramified cover of degree 4 of $\mathbb{P}^1(\mathbb{C})$:

$$C_a \longrightarrow \mathbb{P}^1(\mathbb{C}) \quad (x, y, z) \longrightarrow (x, z).$$

Let $U$ be the open subset of $\mathbb{P}^1(\mathbb{C})$ obtained by cutting the segment joining the points $(-a, 0), (a, 0)$, and let $V = \pi^{-1}(U)$. Since $U$ is simply connected, $V$ must be a connected sum of four sheets homeomorphic to $U$. The four roots of $(x^2 - a^2z^2)(x^2 - a^{-2}z^2)$ provide continuous sections of the map $\pi$ along $U$, which map $U$ homeomorphically onto each one of the sheets of $V$. Put $\alpha_1(t) = \text{arg}(t + a), \alpha_2(t) = \text{arg}(t + 1/a), \alpha_3(t) = \text{arg}(t - 1/a), \alpha_4(t) = \text{arg}(t - a)$, and let $s_k(t) = (t, y_k(t))$, where

$$y_1(t) = \sqrt[4]{|t^2 - a^2| |t^2 - a^{-2}|} \exp i(\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t))/4,$$

$$y_2(t) = iy_1(t),$$

$$y_3(t) = -y_1(t),$$

$$y_4(t) = -iy_1(t).$$
We will use \( s_k(U) \) to refer to the \( k \)-th sheet of the covering. Using the sections \( s_k \), we can lift paths on \( \mathbb{P}^1(\mathbb{C}) \) to paths on \( C_a \). We consider the following paths on \( \mathbb{P}^1(\mathbb{C}) \):

![Diagram of paths on \( \mathbb{P}^1(\mathbb{C}) \)]

and their liftings on \( C_a \) (the numbers indicate the sheet on which each part of the path is located, that is, which of the sections \( s_k \) is used to lift each part of the path):

![Diagram of liftings on \( C_a \)]

It is clear that the paths \( F \) and \( G \) intersect at just one point, on sheet 1. We transport the natural orientation of \( \mathbb{P}^1(\mathbb{C}) \) to \( C_a \) via the map \( \pi \), to ensure that \( (F,G) = +1 \). The automorphisms of \( C_a \) act on \( H_1(C_a,\mathbb{Z}) \). In particular, the automorphism \( \alpha \) lifts the paths on \( C_a \) by one sheet. We use this action to compute a basis for the homology of \( C_a \).

**Proposition 3.1.** The homology classes of the paths

\[
e_1 = F, \quad e_2 = \alpha^2(F), \quad e_3 = \alpha(G) + \alpha^2(G) + F - \alpha^2(F),
\]
\[
e_4 = G, \quad e_5 = \alpha^2(G), \quad e_6 = \alpha^2(G) - G + \alpha(F),
\]

yield a symplectic basis for \( H_1(C_a,\mathbb{Z}) \).

**Proof:** First of all, we assert that \( u_1 = F, u_2 = \alpha(F), u_3 = \alpha^2(F), u_4 = G, u_5 = \alpha(G), \) \( u_6 = \alpha^2(G) \) are a \( \mathbb{Z} \)-basis of \( H_1(C_a,\mathbb{Z}) \). This is seen by determining the intersection matrix of these paths:

\[
\begin{pmatrix}
0 & -1 & 0 & -1 & 0 & 0 \\
1 & 0 & -1 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 \\
1 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]
This matrix has determinant 1, and hence $H_1(C_a, \mathbb{Z}) = < u_1, u_2, u_3, u_4, u_5, u_6 >$. The basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is obtained from the previous one applying the Gram-Schmidt orthogonalization process. \square

4 Differential forms and abelian integrals

**Proposition 4.1.** Let $x = X/Z, y = Y/Z$. The differential forms $\omega_1 = \frac{dx}{y^3}, \omega_2 = \frac{xdx}{y^3}$, $\omega_3 = \frac{dx}{y^2}$ yield an orthogonal basis for the space $H^0(C_a, \Omega^1)$ of abelian differentials of the first kind on $C_a$.

**Proof:** The only non-immediate point is the orthogonality, which is proved using the action of $Aut(C_a)$ on $H^0(C_a, \Omega^1)$. For instance,

$$< \omega_1, \omega_3 > = \frac{i}{2} \int_{C_a} \omega_1 \wedge \overline{\omega_3} = \frac{i}{2} \int_{\alpha(C_a)} \alpha^* (\omega_1 \wedge \overline{\omega_3}) = \frac{i}{2} \int_{C_a} (-i) \omega_1 \wedge (-1) \overline{\omega_3} = i < \omega_1, \omega_3 > \iff < \omega_1, \omega_3 > = 0. \square$$

We will express the period matrix of $C_a$ with respect to this basis in terms of the basic periods

$$f_i := \int_F \omega_i, \quad g_i := \int_G \omega_i.$$ 

Using the automorphisms of $C_a$, we easily find that $f_2 = 0, g_2 = g_1$ and

$$\int_{e_2} \omega_1 = -f_1, \quad \int_{e_2} \omega_2 = 0, \quad \int_{e_2} \omega_3 = f_3,$$

$$\int_{e_3} \omega_1 = 2f_1 + (i - 1)g_1, \quad \int_{e_3} \omega_2 = (i - 1)g_1, \quad \int_{e_3} \omega_3 = 0,$$

$$\int_{e_5} \omega_1 = -g_1, \quad \int_{e_5} \omega_2 = -g_1, \quad \int_{e_5} \omega_3 = g_3,$$

$$\int_{e_6} \omega_1 = (1 + i)f_1 - 2g_1, \quad \int_{e_6} \omega_2 = -2g_1, \quad \int_{e_6} \omega_3 = 0.$$

Since $0 = < \omega_1, \omega_2 > = (2 - 2i)(1 + i)g_1 - i f_1 g_2$, we have $g_1 = \frac{1 + i}{2} f_1$. Hence, the period matrix of $C_a$ has the shape

$$\Omega_a = (\Omega_1 | \Omega_2) = \begin{pmatrix}
  f_1 & -f_1 & \frac{1 + i}{2} f_1 & -\frac{1 + i}{2} f_1 & 0 \\
  0 & 0 & -f_1 & \frac{1 + i}{2} f_1 & -\frac{1 + i}{2} f_1 \\
  f_3 & f_3 & g_3 & g_3 & 0
\end{pmatrix}.$$
Multiplying by \( \Omega_1 \) on the left, we obtain a normalized period matrix \( (I|Z_a) \) for \( C_a \) with respect to the basis \((\eta_1, \eta_2, \eta_3)^t = \Omega_1^{-1}(\omega_1, \omega_2, \omega_3)^t\):

\[
Z_a = \begin{pmatrix}
\frac{g_3}{2f_3} + \frac{1+i}{2} & \frac{g_3}{2f_3} - \frac{1+i}{2} & -\frac{1+i}{2} \\
\frac{g_3}{2f_3} - \frac{1+i}{2} & \frac{g_3}{2f_3} + \frac{1+i}{2} & 1+i \\
-\frac{1+i}{2} & \frac{1+i}{2} & 1+i
\end{pmatrix}.
\]

We thus see that the computation of the abelian integrals on \( C_a \) requires only the computation of the integrals of the form \( \int \omega_3 = \int \frac{dx}{\sqrt{x^4 - (a^2 + a^{-2})x^2 + 1}} \) which are elliptic integrals. The computation of the periods \( f_3 \) and \( g_3 \) is an exercise of elliptic calculus ([W-W 92]). We obtain:

\[
f_3 = -\frac{4}{a}K(a^{-2}), \quad g_3 = -\frac{4i}{a + a^{-1}}K\left(\frac{a - a^{-1}}{a + a^{-1}}\right),
\]

where \( K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \) is the complete elliptic integral of the first kind. In order to make explicit a normalized basis for \( H^0(C_a, \Omega^1) \), we also need the period \( f_1 \), given by:

\[
f_1 = (1+i)\frac{\Gamma\left(\frac{1}{2}\right)^2}{\sqrt{2\pi \sqrt{a^2 - a^{-2}}}}.
\]

**Corollary 4.2.** The differential forms \( \eta_1 = \frac{1}{2f_1}(\omega_1 + \omega_2) + \frac{1}{2f_3}\omega_3 \), \( \eta_2 = -\frac{1}{2f_1}(\omega_1 + \omega_2) + \frac{1}{2f_3}\omega_3 \), \( \eta_3 = \frac{1}{f_1}\omega_2 \) yield a normalized basis for \( H^0(C_a, \Omega^1) \). The corresponding period matrix with respect to the homology basis given in proposition 3.1 is \((I|Z_a)\), where

\[
Z_a = \begin{pmatrix}
\frac{1+i}{2} + i\tau_a & \frac{1+i}{2} + i\tau_a & -\frac{1+i}{2} \\
-\frac{1+i}{2} + i\tau_a & \frac{1+i}{2} + i\tau_a & 1+i \\
-\frac{1+i}{2} & \frac{1+i}{2} & 1+i
\end{pmatrix},
\]

and \( \tau_a = \frac{a^2}{2(a^2 + 1)} \frac{K\left(\frac{a^{-1/a}}{a + a^{-1/a}}\right)}{K(a^{-2})} \).
For instance, for \( a = 1 + \sqrt{2} \) and using the relation 
\[
\frac{1 + k}{2} K(\sqrt{1 - k^2}) = K(\frac{1 - k}{1 + k}),
\]
we obtain \( \tau_{1+\sqrt{2}} = 1/2 \); we thus recover the period lattice of the Fermat curve of fourth degree ([Rh 78]):
\[
Z_{1+\sqrt{2}} = \begin{pmatrix}
\frac{1}{2} + i & -\frac{1}{2} & -\frac{1}{2} - i \\
-\frac{1}{2} & \frac{1}{2} + i & \frac{1}{2} + i \\
-\frac{1}{2} - i & \frac{1}{2} + i & 1 + i
\end{pmatrix}.
\] (3)

5 Theta functions and automorphisms

Let
\[
\theta_a(z) = \theta(z, Z_a) = \sum_{n \in \mathbb{Z}^3} \exp(\pi in^t Z_a n + 2\pi in^t z)
\]
be the theta function associated to the period matrix \( Z_a \) of \( C_a \). The automorphisms of \( C_a \) induce automorphisms on \( J(C_a) \), which we will denote by the same letters. Some of these automorphisms leave \( \theta_a \) invariant:

**Proposition 5.1.** Let \( H \) be the subgroup of \( \text{Aut}(J(C_a)) \) generated by \( \alpha^2, \beta, \gamma \). For every \( \sigma \in H \) we have
\[
\theta_a(\sigma z) = \theta_a(z).
\]

**Proof:** Let \( \Omega = (\Omega_1|\Omega_2) \) be the period matrix of the differential forms \( \omega_i \) with respect to the paths \( e_i \). Given \( \sigma \in H \), let us denote by \( A_\sigma \) the matrix of the automorphism induced by \( \sigma \) in \( H^0(C_a, \Omega^1)^* \) with respect to the dual basis \( \{\omega_1^*, \omega_2^*, \omega_3^*\} \), and by \( M_\sigma \) the matrix of the automorphism induced on \( H_1(C_a, \mathbb{Z}) \) with respect to the basis \( \{e_1, e_2, e_3, e_4, e_5, e_6\} \). In fact, \( A_\sigma \) is the complex representation of \( \sigma \), and \( M_\sigma \) is the rational representation, which is the direct sum of the complex representation and its conjugate. As such, they satisfy the relation
\[
\begin{pmatrix}
A_\sigma & 0 \\
0 & A_\sigma
\end{pmatrix}
\begin{pmatrix}
\Omega \\
\Omega^t
\end{pmatrix}
= 
\begin{pmatrix}
\Omega \\
\Omega^t
\end{pmatrix} M_\sigma.
\]
As we can compute \( A_\sigma \) very easily from the action of \( \text{Aut}(C_a) \) on \( H^0(C_a, \Omega^1) \), we can
determine the matrices $M_\sigma$ with this relation. We obtain:

$$M_{a^2} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}, \quad M_\beta = \begin{pmatrix}
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
\end{pmatrix},$$

$$M_\gamma = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.$$  

Moreover, we can prove that the non-zero blocks composing the matrices $M_\sigma$ are the matrices $B_\sigma = \Omega_1^{-1} A_\sigma \Omega_1$ of the complex representation of $\sigma$ with respect to the normalized basis of $H^0(\mathcal{C}_a, \Omega_1)$ given in proposition 3.1.

We now take into account the modular behaviour of the theta function: given a symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that the products $A^t C$ and $B^t D$ have even diagonals, the theta function satisfies (cf. [Mu 82]):

$$\theta((CZ + D)^{i-1} z, (AZ + B)(CZ + D)^{-1}) = \zeta_S \sqrt{\det(CZ + D)} e^{(\pi i z ((CZ + D)^{-1} Cz))} \theta(z, Z),$$

where $\zeta_S$ is an 8-th root of the unity, depending only on $S$. For our matrices, this relation reduces to

$$\theta(A z, AZA^t) = \zeta_S \theta(z, Z).$$

Moreover, as $\det A = 1$, a change of variables in the sum defining $\theta$ yields $\theta(u, A.Z.A^t) = \theta(A.u, Z)$. Substituting $u = \sigma(z) = A.z$ in this relation and noting that our matrices satisfy also $A^2 = I$, we obtain $\theta(A.z, Z) = \zeta_S \theta(z, Z)$. Taking $z = 0$ we see that $\zeta_S = 1$ for any of our matrices. □

We differentiate this formula with respect to the variables $z_i$ to obtain:

**Corollary 5.2.** Let $B_\sigma = \Omega_1^{-1}. A_\sigma. \Omega_1$ the complex representation of $\sigma$ with respect to the normalized basis of $H^0(\mathcal{C}_a, \Omega_1)$ given in proposition 3.1. For any $\sigma \in H$:

$$\left( \frac{\partial \theta_a}{\partial z_1}(\sigma z), \frac{\partial \theta_a}{\partial z_2}(\sigma z), \frac{\partial \theta_a}{\partial z_3}(\sigma z) \right) = \left( \frac{\partial \theta_a}{\partial z_1}(z), \frac{\partial \theta_a}{\partial z_2}(z), \frac{\partial \theta_a}{\partial z_3}(z) \right) B_\sigma.$$
6 Half-periods and bitangent lines

It is well known ([ACGH 85]) that for genus three curves there is a bijection between the 28 bitangent lines and the 28 odd 2-torsion points of \( J(C_a) \). This bijection, induced by the Abel-Jacobi map, goes like this: given an odd 2-torsion point \( z_0 \) on \( J(C_a) \), its corresponding bitangent line is:

\[
\left( \frac{\partial \theta_a}{\partial z_1}(z_0), \frac{\partial \theta_a}{\partial z_2}(z_0), \frac{\partial \theta_a}{\partial z_3}(z_0) \right) \Omega_1^{-1} \begin{pmatrix} Z \\ X \\ Y \end{pmatrix} = 0.
\]

We will make this bijection completely explicit for our curves \( C_a \). Let us enumerate the 2-torsion points of \( J(C_a) \). Put \( m_1 = (0, 0, 0)^t \), \( m_2 = (0, 0, 1)^t \), \( m_3 = (0, 1, 0)^t \), \( m_4 = (0, 1, 1)^t \), \( m_5 = (1, 0, 0)^t \), \( m_6 = (1, 0, 1)^t \), \( m_7 = (1, 1, 0)^t \), \( m_8 = (1, 1, 1)^t \), and let \( z_{jk} := \frac{1}{2}(m_j + Z a m_k) \). The 2-torsion point \( z_{jk} \) is odd if and only if \( m_j^t m_k \equiv 1 \mod 2 \).

The first step in the construction of the bijection is the comparison of the action of the automorphisms of \( C_a \) on the 2-torsion points and their action on the bitangent lines. The action on the 2-torsion points is given by the matrices \( B_\sigma \). The action on the bitangent lines can be computed directly, and is summarized in the following table (subindexes \( r \) must be considered \( \mod 8 \), and \( s \) must be considered \( \mod 4 \)):

<table>
<thead>
<tr>
<th></th>
<th>( t_a )</th>
<th>( t_{-a} )</th>
<th>( t_{1/a} )</th>
<th>( t_{-1/a} )</th>
<th>( L_r )</th>
<th>( L_r^t )</th>
<th>( \ell_{0,s} )</th>
<th>( \ell_{1,s} )</th>
<th>( \ell_{2,s} )</th>
<th>( \ell_{3,s} )</th>
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<td>( \alpha )</td>
<td>( t_a )</td>
<td>( t_{-a} )</td>
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<td>( t_{-1/a} )</td>
<td>( L_r )</td>
<td>( L_r^t )</td>
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<td>( \ell_{3,s} )</td>
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<td>( t_{1/a} )</td>
<td>( L_{r+1} )</td>
<td>( L_{r+1}^t )</td>
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<td>( t_{1/a} )</td>
<td>( t_{-1/a} )</td>
<td>( t_{a} )</td>
<td>( t_{-a} )</td>
<td>( L_r )</td>
<td>( L_r^t )</td>
<td>( \ell_{0,s} )</td>
<td>( \ell_{3,s+1} )</td>
<td>( \ell_{2,s+2} )</td>
<td>( \ell_{1,s-1} )</td>
</tr>
</tbody>
</table>

The quadritangent lines \( t_{\pm a}, t_{\pm 1/a} \) are the only bitangent lines fixed by \( \alpha^2 \). The only odd 2-torsion points fixed by \( \alpha^2 \) are \( z_{22}, z_{28}, z_{82}, z_{88} \), so they must correspond to the quadritangent lines. Let us take one of these 2-torsion points, \( z_{22} \) for instance. Its corresponding bitangent line

\[
\left( \frac{\partial \theta_a}{\partial z_1}(z_{22}) - \frac{\partial \theta_a}{\partial z_2}(z_{22}) \right) Z + \left( \frac{\partial \theta_a}{\partial z_1}(z_{22}) \right) X - \left( \frac{\partial \theta_a}{\partial z_3}(z_{22}) \right) Y = 0
\]

must coincide with one of the equations of \( t_{\pm a \pm 1}, X = \pm a \pm 1 Z \). So, we must have \( \frac{\partial \theta_a}{\partial z_1}(z_{22}) = -\frac{\partial \theta_a}{\partial z_2}(z_{22}) \), and

\[
\frac{\partial \theta_a}{\partial z_3}(z_{22}) \in \{1 \pm a^{\pm 1} \}.
\]
Add $\pm a^{\pm 1}$ to this quotient, and consider these expressions as functions of $a$. As such, they are continuous functions; exactly one of them takes the value 1 and it is constant. A numerical calculation of them for any value of $a$ will tell us which bitangent line corresponds to $z_{22}$. We can make the computations for the value $a = 1 + \sqrt{2}$, for which we know the exact value of the period matrix $Z_a$ (cf. 3). We have to sum the triple series

$$\frac{\partial \theta_a}{\partial z_j}(z_{22}) = 2\pi i \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \sum_{n_3 \in \mathbb{Z}} n_j \exp(\pi i (n_1, n_2, n_3)^t Z(n_1, n_2, n_3) + 2\pi i (n_1, n_2, n_3).z_{22}).$$

In practice, it is enough to sum about one thousand terms of these triple series. One obtains

$$\frac{\partial \theta_a}{\partial z_3}(z_{22}) = 2 + \sqrt{2} = a + 1,$$

so that the bitangent line corresponding to the $2$-torsion point $z_{22}$ is $t_{1/a}$.

We can argue similarly to determine the bitangent lines corresponding to the rest of odd $2$-torsion points of $J(C_a)$. Of course, corollary 5.2 saves us a great deal of work. For instance, as $\gamma(z_{22}) = z_{28}$, we can ensure directly that the bitangent line corresponding to $z_{28}$ is $t_a$.

**Theorem 6.1.** The one-to-one correspondence between bitangent lines to the curve $C_a$ and the odd $2$-torsion points of $J(C_a)$, induced by the Abel-Jacobi map, is given by:

<table>
<thead>
<tr>
<th>$t_a$</th>
<th>$z_{28}$</th>
<th>$t_{-a}$</th>
<th>$z_{38}$</th>
<th>$t_{1/a}$</th>
<th>$z_{22}$</th>
<th>$t_{-1/a}$</th>
<th>$z_{82}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^1$</td>
<td>$z_{57}$</td>
<td>$L^3$</td>
<td>$z_{38}$</td>
<td>$L^0$</td>
<td>$z_{22}$</td>
<td>$L'$</td>
<td>$z_{58}$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>$z_{67}$</td>
<td>$L_3$</td>
<td>$z_{42}$</td>
<td>$L_5$</td>
<td>$z_{47}$</td>
<td>$L_7$</td>
<td>$z_{62}$</td>
</tr>
<tr>
<td>$\ell_{00}$</td>
<td>$z_{85}$</td>
<td>$\ell_{01}$</td>
<td>$z_{73}$</td>
<td>$\ell_{02}$</td>
<td>$z_{75}$</td>
<td>$\ell_{03}$</td>
<td>$z_{83}$</td>
</tr>
<tr>
<td>$\ell_{10}$</td>
<td>$z_{64}$</td>
<td>$\ell_{11}$</td>
<td>$z_{33}$</td>
<td>$\ell_{12}$</td>
<td>$z_{46}$</td>
<td>$\ell_{13}$</td>
<td>$z_{55}$</td>
</tr>
<tr>
<td>$\ell_{20}$</td>
<td>$z_{24}$</td>
<td>$\ell_{21}$</td>
<td>$z_{76}$</td>
<td>$\ell_{22}$</td>
<td>$z_{26}$</td>
<td>$\ell_{23}$</td>
<td>$z_{74}$</td>
</tr>
<tr>
<td>$\ell_{30}$</td>
<td>$z_{65}$</td>
<td>$\ell_{31}$</td>
<td>$z_{56}$</td>
<td>$\ell_{32}$</td>
<td>$z_{43}$</td>
<td>$\ell_{33}$</td>
<td>$z_{34}$</td>
</tr>
</tbody>
</table>

7 Splitting of the jacobian

In section 4 we have reduced the computation of the period matrix of $C_a$ to *elliptic calculus*. We could say that all the differential forms of the first kind on $C_a$ are elliptic, but this amounts to saying that the jacobian $J(C_a)$ splits as a product of elliptic curves. We will see in a moment that this is the case. The point is to interpret the changes of variables used in the computation of the periods of $C_a$ as maps from $C_a$ to elliptic curves.
The following propositions involve straightforward computations:

**Proposition 7.1.** Let $E_a$ be the genus one curve given by

$$Y^2Z^2 = (X^2 - a^2Z^2)(X^2 - a^{-2}Z^2).$$

i) The degree 2 map

$$\psi_a : \mathcal{C}_a \longrightarrow E_a$$

$$(x, y, z) \longrightarrow (xz, y^2, z^2),$$

is the quotient map from $\mathcal{C}_a$ to $\mathcal{C}_a/ \langle \alpha^2 \rangle$.

ii) Let $u = X/Z, v = Y/Z$. The differential form $\omega_a = \frac{du}{v}$ associated to $E_a$ satisfies

$$\psi_a^*(\omega_a) = \omega_3.$$

iii) The period lattice of $E_a$ is $\Lambda_a = \langle f_3, g_3 \rangle$.

iv) A Weierstrass equation for $E_a$ is $Y^2Z = X(X - Z)(X - nZ)$, where $n = \left(\frac{a + a^{-1}}{2}\right)^2$.

**Proposition 7.2.** Let $E$ be the elliptic curve given by $Y^2Z = X^3 - XZ^2$. Let us write

$$\mu = \sqrt{\frac{a^2 - a^{-2}}{2}}, \quad m = \frac{a^2 + a^{-2}}{2}, \quad \zeta = \frac{1 + i}{\sqrt{2}}.$$

i) The degree 2 map

$$\psi_2 : \mathcal{C}_a \longrightarrow E$$

$$(x, y, z) \longrightarrow (i(y^2 - \zeta^2\mu^2z^2), (i - 1)(x^2 - mz^2), (y - \zeta mz)^2),$$

is the quotient map from $\mathcal{C}_a$ to $\mathcal{C}_a/ \langle \beta \rangle$.

ii) Let $u = X/Z, v = Y/Z$. The differential form $\omega_E = \frac{du}{2v}$ associated to $E$ satisfies

$$\psi_2^*(\omega_E) = -\left(\mu/\sqrt{2}\right)\omega_2$$

iii) The period lattice of $E$ is $\Lambda_E = \langle \tau, i\tau \rangle$, with $\tau = \frac{\Gamma(\frac{1}{4})^2}{\sqrt{2\pi}} = \zeta^{-1}f_1$.

iv) The map $\psi_1 = \psi_2 \circ \gamma$

$$\psi_1 : \mathcal{C}_a \longrightarrow E$$

$$(x, y, z) \longrightarrow (i(y^2 - \zeta^2\mu^2x^2), (i - 1)(z^2 - mx^2), (y - \zeta mx)^2),$$

has analogous properties. In particular, $\psi_1^*(\omega_E) = \left(\mu/\sqrt{2}\right)\omega_1$. 

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The maps $\psi_0, \psi_1, \psi_2$ induce maps from $J(C_a)$ to the elliptic curves $E_a$ and $E$, which we will denote by $\Psi_0, \Psi_1, \Psi_2$ respectively.

**Theorem 7.3.** Let $\Psi$ be the product map $\Psi = (\Psi_1, \Psi_2, \Psi_3) : J(C_a) \rightarrow E \times E \times E_a$.

a) The map $\psi$ is a degree 8 isogeny, defined over $\mathbb{Q}(a, \zeta_8)$. Its kernel is a 2-torsion subgroup of $J(C_a)$, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

b) Let $D^r$ (resp. $D_r, D_{rs}$) be the divisor given by the contact points of the bitangent line $L^r$ (resp. $L_r, L_{rs}$) with the curve $C_a$, and let $P_{\pm a \pm 1}$ be the contact point of the quadritangent line $t_{\pm a \pm 1}$. The kernel of $\Psi$ is generated by the classes of the divisors $2P_a - 2P_a$, $D^1 - D^7$, $D_{00} - D_{03}$.

**Proof:** We will find the complex representation and the rational representation of the map $\Psi$. We fix $\{\omega_1, \omega_2, \omega_3\}$ as basis of $H^0(C_a, \Omega^1)$, $\omega_E$ as basis of $H^0(E, \Omega^1)$, and $\omega_a$ as basis of $H^0(E_a, \Omega^1)$. Using the preceding propositions, we see that the complex representation of $\Psi$ in terms of these bases is

$$\Psi \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} (\mu/\sqrt{2})z_1 \\ -(\mu/\sqrt{2})z_2 \\ z_3 \end{pmatrix}.$$  (4)

Let $e_1, e_2, e_3, e_4, e_5, e_6$ be the columns of the period matrix $\Omega_a$. We fix these vectors as a basis for the period lattice of $C_a$, and choose

$$u_1 = \begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} i\tau \\ 0 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} \tau \\ 0 \\ 0 \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ i\tau \\ 0 \end{pmatrix}, u_5 = \begin{pmatrix} 0 \\ 0 \\ f_3 \end{pmatrix}, u_6 = \begin{pmatrix} 0 \\ 0 \\ g_3 \end{pmatrix},$$

as basis for the product lattice $\Lambda_E \times \Lambda_E \times \Lambda_a$. Let $v_i = \Psi(e_i)$. Using (4), we find that

$$v_1 = u_1 + u_2 + u_5 \quad v_2 = -u_1 - u_2 + u_5 \quad v_3 = u_1 + u_2 + u_5 + u_4 \quad v_4 = u_2 - u_4 + u_6 \quad v_5 = -u_2 + u_4 + u_5 \quad v_6 = 2u_4.$$  

Thus, if we consider the basis $w_1 = u_1, w_2 = u_2 - u_1, w_3 = u_3, w_4 = -u_1 + u_2 + u_4 - u_3, w_5 = -u_1 + u_3 + u_5, w_6 = u_1 - u_2 + u_6$ for $\Lambda_E \times \Lambda_E \times \Lambda_a$, and the basis $t_1 = v_1, t_2 = v_4, t_3 = v_3 - v_1, t_4 = v_6, t_5 = v_1 + v_2, t_6 = v_4 + v_5$ for the period lattice of $C_a$, we see that the corresponding rational representation of $\Psi$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$
It is evident that the kernel of $\Psi$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and that it is generated by the divisors corresponding to the 2-torsion points $-\frac{1}{2}\Psi^{-1}(v_6) = -\frac{1}{2}e_6$, $-\frac{1}{2}\Psi^{-1}(v_1 + v_2) = -\frac{1}{2}(e_1 + e_2)$, $-\frac{1}{2}\Psi^{-1}(v_4 + v_5) = -\frac{1}{2}(e_4 + e_5)$. Expressing these vectors in the normalized basis of $H^0(C_a, \Omega^1)$, we see that they are the 2-torsion points $z_{5,7} - z_{5,8}, z_{88} - z_{28}, z_{83} - z_{83}$. The table in theorem 6.1 gives the divisors corresponding to these points.\[\Box\]

**Remark:** It was already known that the jacobian of the Fermat curve of fourth degree $F_4$ is 64-isogenous over $\mathbb{Q}$ to a product of three copies of the elliptic curve $E$ ([Li 92]). Our computations provide a new isogeny defined over $\mathbb{Q}(\sqrt{2})$, which has degree 8. The reason for this difference is that our quotients of $F_4$ are finer than the classical quotients. For instance, we consider $F_4/\langle \alpha^2 \rangle$ instead of $F_4/\langle \alpha \rangle$, which was one of the already known quotients of $F_4$.

**References**


[Rh 78] D. Rohrlich, ‘The periods of the Fermat curve’, appendix to [Gr 78].