The Frobenius Problem: A Geometric Approach

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The Frobenius problem (a.k.a. the money changing problem)

$A = \{a_1, \ldots, a_n\}$ a set of positive integers

- Which is the maximum positive integer that can not be represented as a positive combination of the elements of A?

There is a solution, denoted by $g(a_1, \ldots, a_n)$, provided that $\gcd(a_1, \ldots, a_n) = 1$.
Example $A = \{5, 7\}$

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 ,25, 26, . . .

Related problems

- The number of non-representable integers
- A description of the set of non-representable integers
- The *denumerant* of a positive integer m: the number of representations of m
- Postage stamp problem
- ...

Notation

$A = \{a_1, \ldots, a_n\}$ s.t. $a_i > 0$ and $\gcd(a_1, \ldots, a_n) = 1$

- Set of representable integers
  $R(A) = R(a_1, \ldots, a_n) = \{m = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \mid x_i \in \mathbb{N}\}$
- Set of non-representable integers , or *gaps*
  $$\overline{R}(A) = \overline{R}(a_1, \ldots, a_n) = \mathbb{N} \setminus R(A)$$
- The Frobenius number of A
  $$g(A) = g(a_1, \ldots, a_n) = \max \overline{R}(A)$$
- Number of gaps
  $$N(A) = N(a_1, \ldots, a_n) = |\overline{R}(A)|$$
- Number of representations of m, the *denumerant* of m
  $$d(m; A)$$

n-dimensional lattice

$A = \{a_1, \ldots, a_n\}$, with $\gcd(a_1, \ldots, a_n) = 1$

$$\begin{array}{rccc} \ell : & \mathbb{Z}^n & \longrightarrow & \mathbb{Z} \\ & (x_1, \ldots, x_n) & \longmapsto & x_1 a_1 + \cdots + x_n a_n \end{array}$$

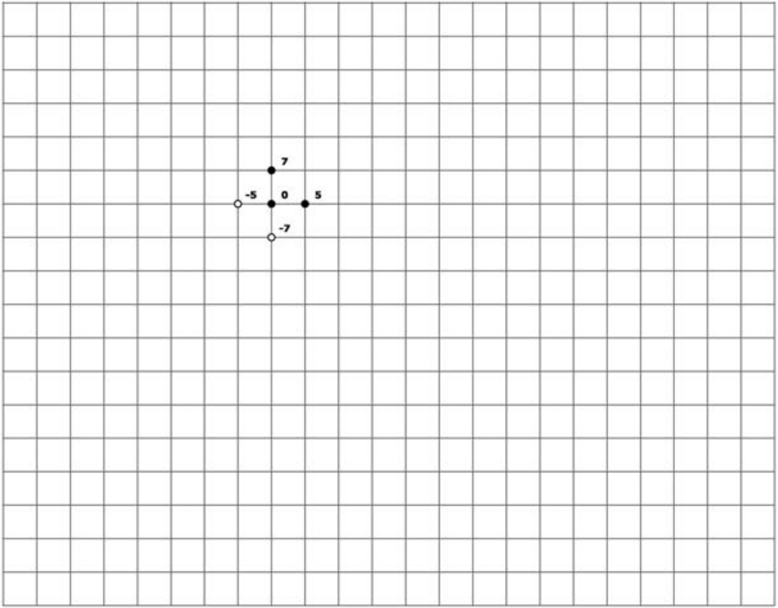
- ℓ is an integer labeling of $\mathbb{Z}^n$. If $m = x_1 a_1 + \cdots x_n a_n$, we label the point of coordinates $(x_1, \ldots, x_n)$ by the integer m.
- ℓ is linear and surjective, and $\ker \ell$ is the set of integer points lying on the hyperplane $x_1 a_1 + \cdots + x_n a_n = 0$

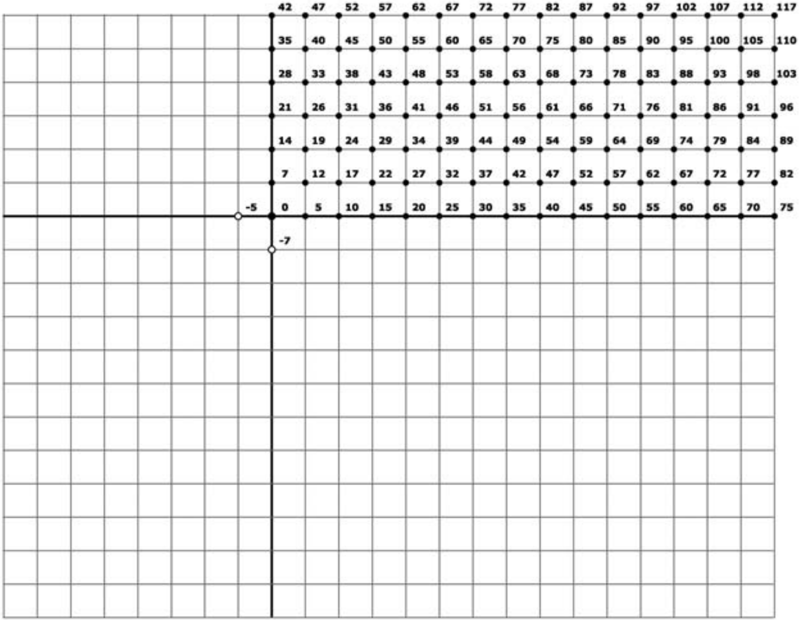
$$\ker \ell = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \,|\, x_1 a_1 + \cdots + x_n a_n = 0\}$$

$A = \{a, b\}$, with $(a, b) = 1$ and $a < b$

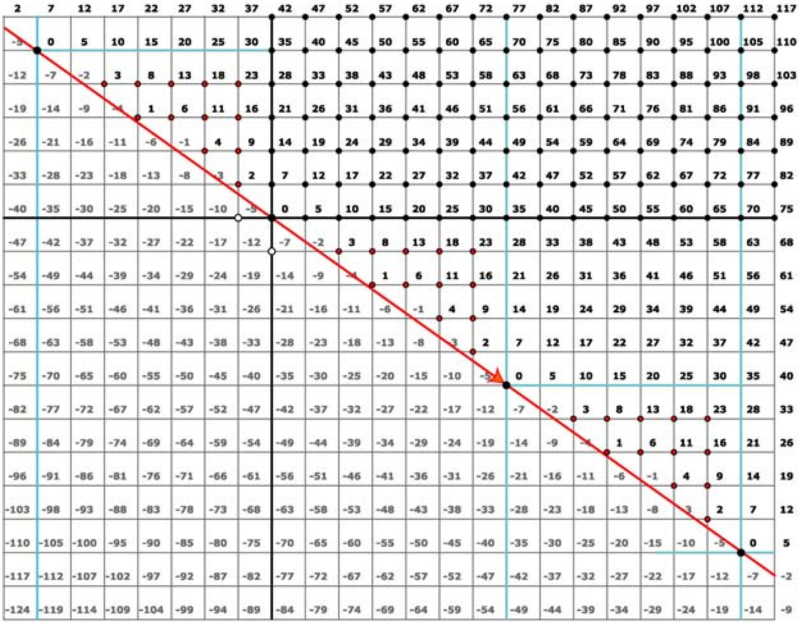
$$\begin{array}{rccc} \ell : & \mathbb{Z}^2 & \rightarrow & \mathbb{Z} \\ & (x, y) & \mapsto & xa + yb \end{array}$$

- $\ell(x, y)$ is the *label* of (x, y)
- $\ker \ell = \{(\lambda b, -\lambda a) \mid \lambda \in \mathbb{Z}\}$ = integer points lying on the line $xa + yb = 0$
- If $m = ua + vb$ then, for every $\lambda \in \mathbb{Z}$, $\ell((u, v) + (\lambda b, -\lambda a)) = m$
- ℓ is periodic with period $(b, -a)$


7
-5
0
5
-7


42 47 52 57 62 67 72 77 82 87 92 97 102 107 112 117
35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110
28 33 38 43 48 53 58 63 68 73 78 83 88 93 98 103
21 26 31 36 41 46 51 56 61 66 71 76 81 86 91 96
14 19 24 29 34 39 44 49 54 59 64 69 74 79 84 89
7 12 17 22 27 32 37 42 47 52 57 62 67 72 77 82
-5 0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75
-7

2	7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82	87	92	97	102	107	112	117
-5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110
-12	-7	-2	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73	78	83	88	93	98	103
-19	-14	-9	-4	1	6	11	16	21	26	31	36	41	46	51	56	61	66	71	76	81	86	91	96
-26	-21	-16	-11	-6	-1	4	9	14	19	24	29	34	39	44	49	54	59	64	69	74	79	84	89
-33	-28	-23	-18	-13	-8	-3	2	7	12	17	22	27	32	37	42	47	52	57	62	67	72	77	82
-40	-35	-30	-25	-20	-15	-10	-5	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
-47	-42	-37	-32	-27	-22	-17	-12	-7	-2	3	8	13	18	23	28	33	38	43	48	53	58	63	68
-54	-49	-44	-39	-34	-29	-24	-19	-14	-9	-4	1	6	11	16	21	26	31	36	41	46	51	56	61
-61	-56	-51	-46	-41	-36	-31	-26	-21	-16	-11	-6	-1	4	9	14	19	24	29	34	39	44	49	54
-68	-63	-58	-53	-48	-43	-38	-33	-28	-23	-18	-13	-8	-3	2	7	12	17	22	27	32	37	42	47
-75	-70	-65	-60	-55	-50	-45	-40	-35	-30	-25	-20	-15	-10	-5	0	5	10	15	20	25	30	35	40
-82	-77	-72	-67	-62	-57	-52	-47	-42	-37	-32	-27	-22	-17	-12	-7	-2	3	8	13	18	23	28	33
-89	-84	-79	-74	-69	-64	-59	-54	-49	-44	-39	-34	-29	-24	-19	-14	-9	-4	1	6	11	16	21	26
-96	-91	-86	-81	-76	-71	-66	-61	-56	-51	-46	-41	-36	-31	-26	-21	-16	-11	-6	-1	4	9	14	19
-103	-98	-93	-88	-83	-78	-73	-68	-63	-58	-53	-48	-43	-38	-33	-28	-23	-18	-13	-8	-3	2	7	12
-110	-105	-100	-95	-90	-85	-80	-75	-70	-65	-60	-55	-50	-45	-40	-35	-30	-25	-20	-15	-10	-5	0	5
-117	-112	-107	-102	-97	-92	-87	-82	-77	-72	-67	-62	-57	-52	-47	-42	-37	-32	-27	-22	-17	-12	-7	-2
-124	-119	-114	-109	-104	-99	-94	-89	-84	-79	-74	-69	-64	-59	-54	-49	-44	-39	-34	-29	-24	-19	-14	-9


2 7 12 17 22 27 32 37 42 47 52 57 62 67 72 77 82 87 92 97 102 107 112 117
-5 0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110
-12 -7 -2 3 8 13 18 23 28 33 38 43 48 53 58 63 68 73 78 83 88 93 98 103
-19 -14 -9 -4 1 6 11 16 21 26 31 36 41 46 51 56 61 66 71 76 81 86 91 96
-26 -21 -16 -11 -6 -1 4 9 14 19 24 29 34 39 44 49 54 59 64 69 74 79 84 89
-33 -28 -23 -18 -13 -8 -3 2 7 12 17 22 27 32 37 42 47 52 57 62 67 72 77 82
-40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75
-47 -42 -37 -32 -27 -22 -17 -12 -7 -2 3 8 13 18 23 28 33 38 43 48 53 58 63 68
-54 -49 -44 -39 -34 -29 -24 -19 -14 -9 -4 1 6 11 16 21 26 31 36 41 46 51 56 61
-61 -56 -51 -46 -41 -36 -31 -26 -21 -16 -11 -6 -1 4 9 14 19 24 29 34 39 44 49 54
-68 -63 -58 -53 -48 -43 -38 -33 -28 -23 -18 -13 -8 -3 2 7 12 17 22 27 32 37 42 47
-75 -70 -65 -60 -55 -50 -45 -40 -35 -30 -25 -20 -15 -10 -5 0 5 10 15 20 25 30 35 40
-82 -77 -72 -67 -62 -57 -52 -47 -42 -37 -32 -27 -22 -17 -12 -7 -2 3 8 13 18 23 28 33
-89 -84 -79 -74 -69 -64 -59 -54 -49 -44 -39 -34 -29 -24 -19 -14 -9 -4 1 6 11 16 21 26
-96 -91 -86 -81 -76 -71 -66 -61 -56 -51 -46 -41 -36 -31 -26 -21 -16 -11 -6 -1 4 9 14 19
-103 -98 -93 -88 -83 -78 -73 -68 -63 -58 -53 -48 -43 -38 -33 -28 -23 -18 -13 -8 -3 2 7 12
-110 -105 -100 -95 -90 -85 -80 -75 -70 -65 -60 -55 -50 -45 -40 -35 -30 -25 -20 -15 -10 -5 0 5
-117 -112 -107 -102 -97 -92 -87 -82 -77 -72 -67 -62 -57 -52 -47 -42 -37 -32 -27 -22 -17 -12 -7 -2
-124 -119 -114 -109 -104 -99 -94 -89 -84 -79 -74 -69 -64 -59 -54 -49 -44 -39 -34 -29 -24 -19 -14 -9

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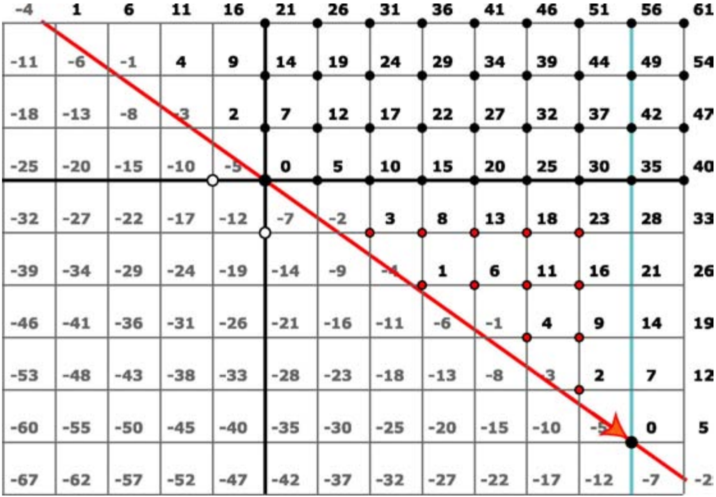
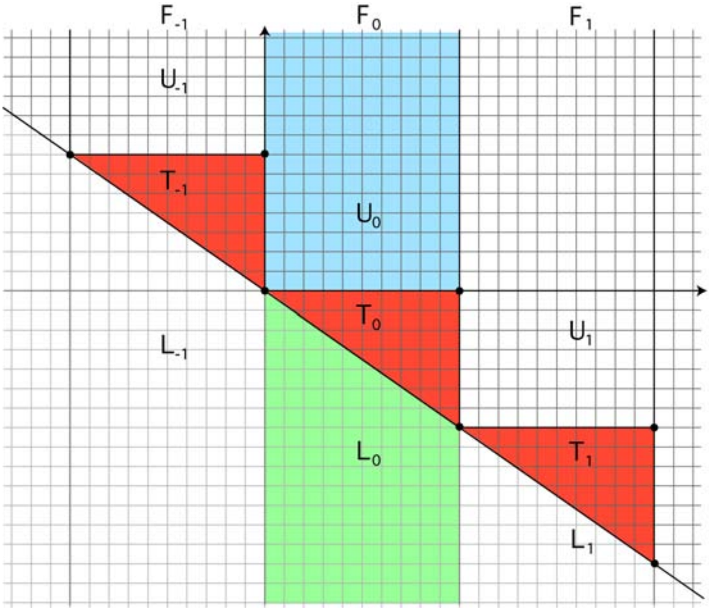



F-1
F0
F1
U-1
T-1
U0
T0
U1
L-1
L0
T1
L1

The lattice partition

$F_\lambda = \{(x, y) \in \mathbb{Z}^2 \,|\, \lambda b \leq x < (\lambda + 1)b\}$
$\ell_{|F_\lambda} : F_\lambda \to \mathbb{Z}$ is a bijection

- Positive integers: above the line $ax + by = 0$

$$U_\lambda = \{(x, y) \in F_\lambda \,|\, ax + by \geq 0,\ y \geq -\lambda a\}$$
$$\ell(U_\lambda) = R(a, b)$$

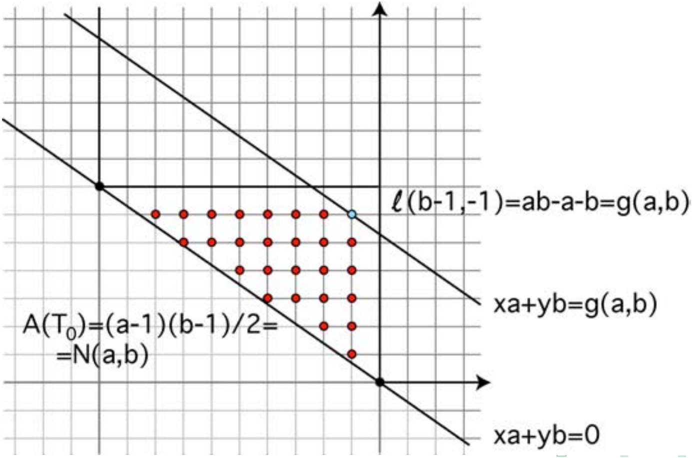
$$T_\lambda = \{(x, y) \in F_\lambda \,|\, ax + by > 0,\ y < -\lambda a\}$$
$$\ell(T_\lambda) = \overline{R}(a, b)$$

- Negative integers: below the line $ax + by = 0$

$$L_\lambda = \{(x, y) \in F_\lambda \,|\, ax + by < 0\}$$

Visualization of known results

- $g(a, b) = \max\{\ell(x, y) | (x, y) \in T_0\} = \ell(b-1, -1) = ab - a - b$
- $N(A) = |\overline{R}(A)| = \frac{(a-1)(b-1)}{2}$



Theorem 1

Let $m \in \mathbb{N}$ and $m = \ell(u, v)$. Then, $d(m; a, b) = \left\lfloor \frac{u}{b} \right\rfloor + \left\lfloor \frac{v}{a} \right\rfloor + 1$

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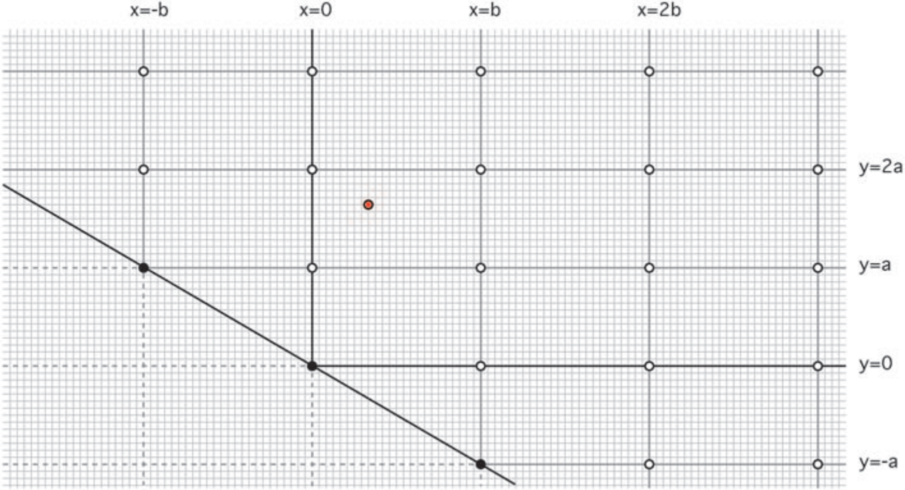

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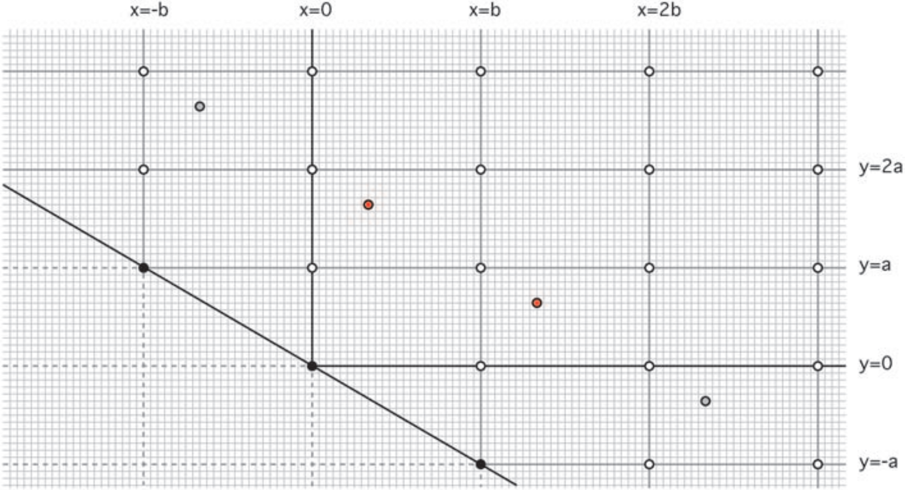

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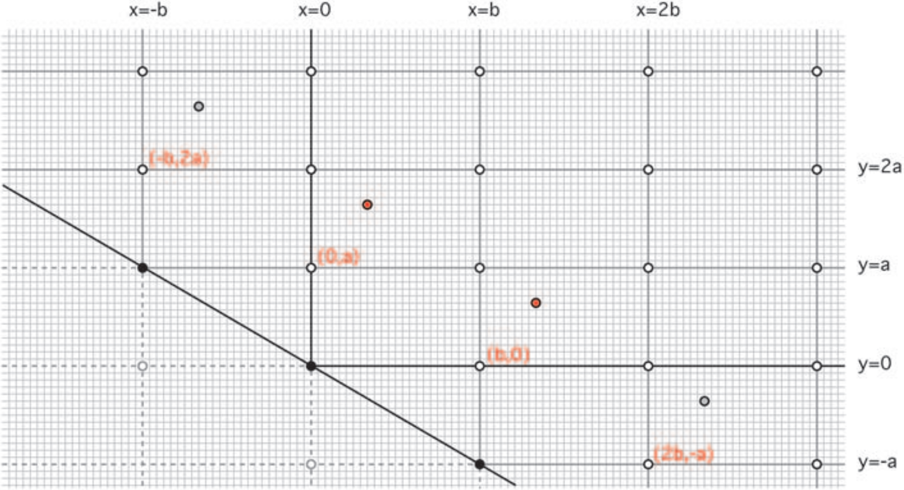
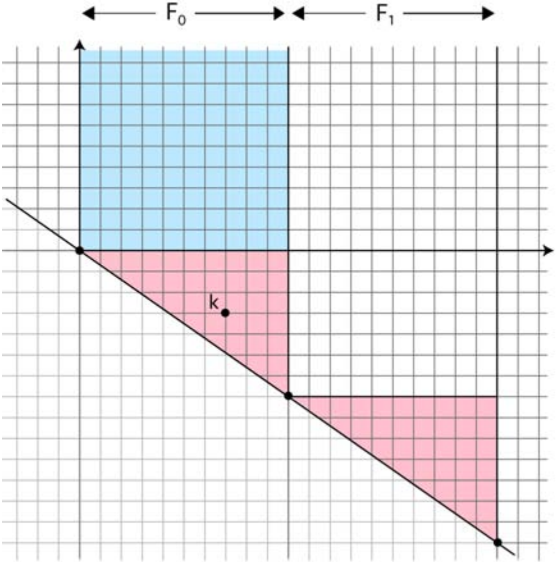


$A = \{a, b, c\}$, with $(a, b) = 1$, $a < b$, and $c \in \overline{R}(a, b)$

$$\begin{array}{rcl} \ell : \qquad \mathbb{Z}^3 & \rightarrow & \mathbb{Z} \\ (x, y, z) & \mapsto & xa + yb + zc \end{array}$$

- $\ell(x, y, z)$ is the *label* of (x, y, z)
- $\ker \ell$ integer points which lie on the plane $xa + yb + zc = 0$
- If $m = ua + vb + wz$ then, $\ell((u, v, w) + (\lambda_1, \lambda_2, \lambda_3)) = m$ for every $(\lambda_1, \lambda_2, \lambda_3)$ in the plane $xa + yb + zc = 0$

Adding k to the set $R(a, b)$



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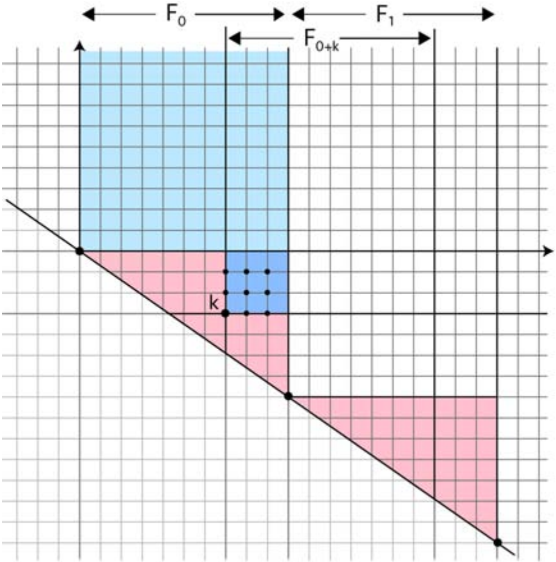

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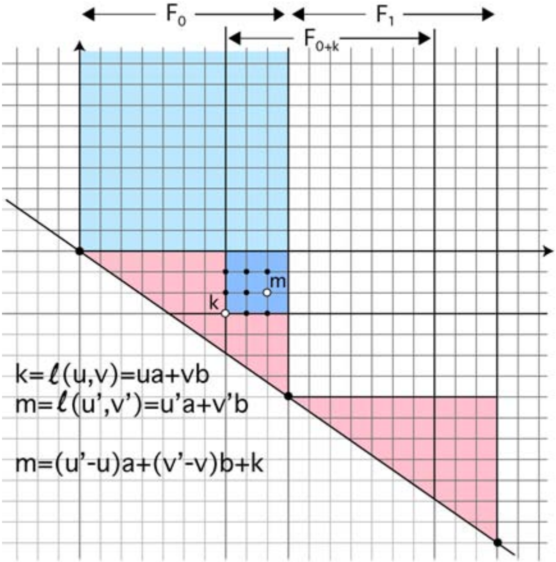


Adding c, $2c$, $3c$, $\ldots$ to the set $R(a, b)$

$k \in \overline{R}(a, b)$

- $R_k(a, b) = \{m = ua + vb + k, u, v \geq 0\} = \ell(U_0 + k)$
- $X_k = (U_0 + k) \cap T_0$
- $\ell(X_k) = R_k(a, b) \setminus R(a, b)$

Property 1

$$R(a, b, c) = \bigcup_{i=0}^{\infty} R_{ic}(a, b)$$

Property 2 The set of points in T_0 whose labels are in $R(a, b, c)$ is

$$\bigcup_{i=0}^{\infty} X_{ic}$$

First representable multiple of $c \in \overline{R}(a, b)$

$c, 2c, \ldots, (s-1)c$ are in $\overline{R}(a, b)$ and sc is in $R(a, b)$

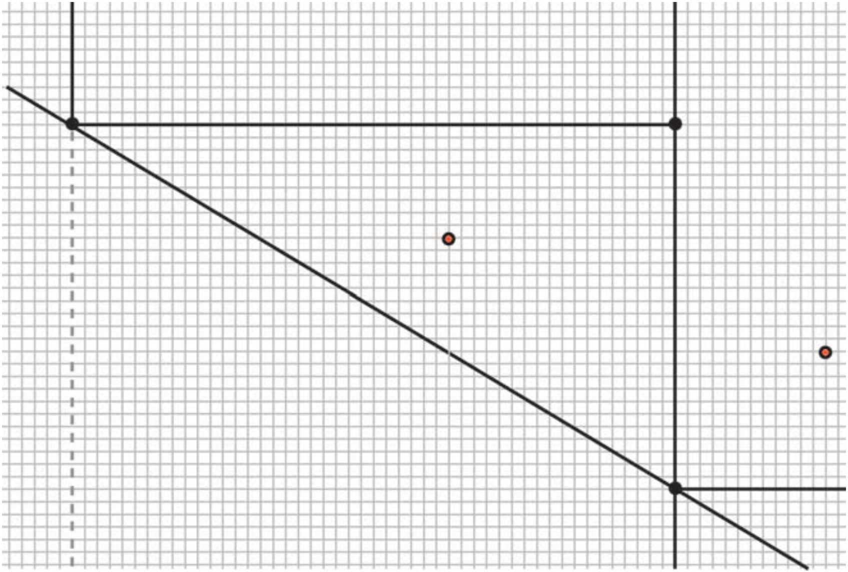
Proposition

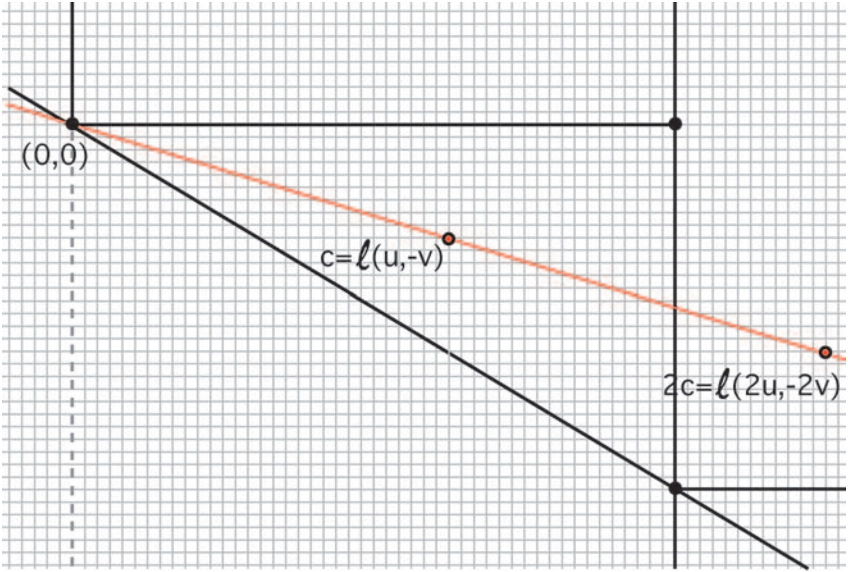
If $c \in \overline{R}(a, b)$ and s is the first representable multiple of c on $\{a, b\}$, then

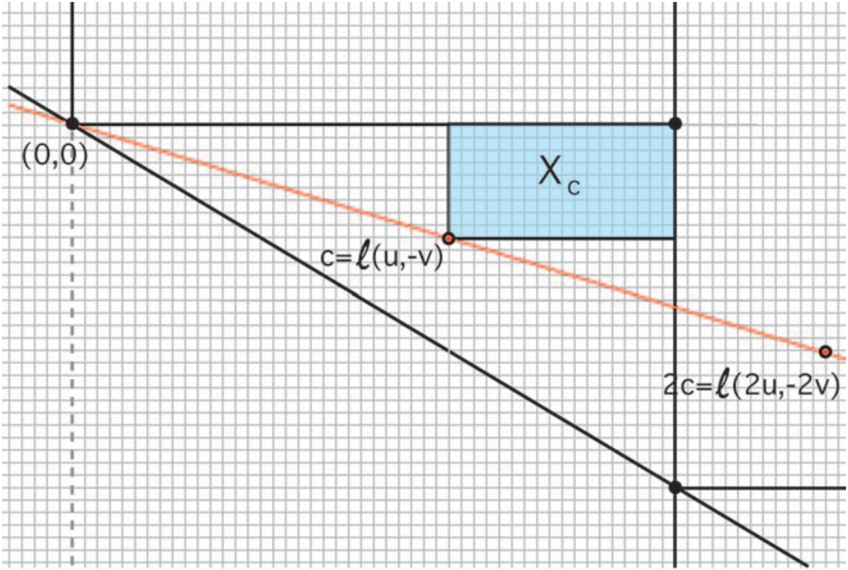
$$R(a, b, c) = \bigcup_{i=0}^{s-1} R_{ic}(a, b)$$

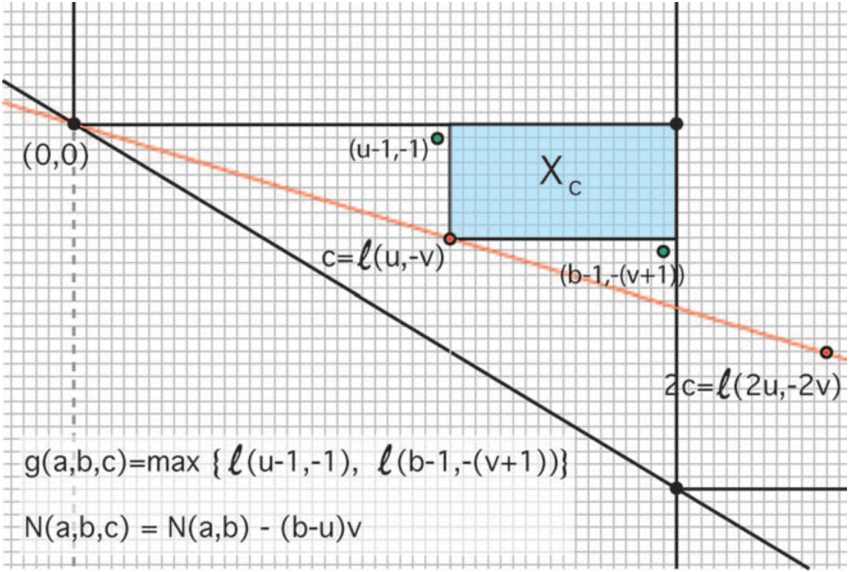
Corollary

$$\overline{R}(a, b, c) = \overline{R}(a, b) \setminus \bigcup_{i=1}^{s-1} \ell(X_{ic})$$




(0,0)
c=ℓ(u,-v)
2c=ℓ(2u,-2v)

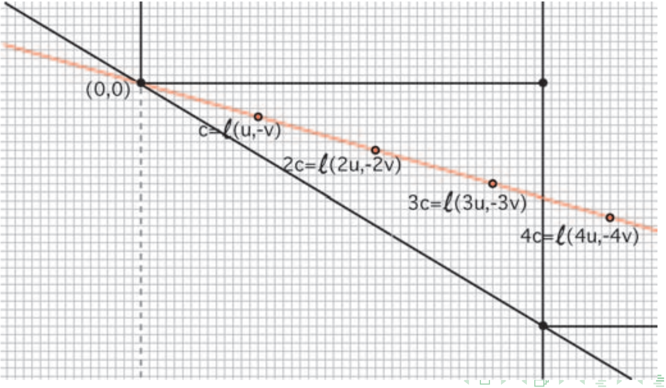

(0,0)
X c
c=ℓ(u,-v)
2c=ℓ(2u,-2v)


(0,0)
(u-1,-1)
X c
c=ℓ(u,-v)
(b-1,-(v+1))
2c=ℓ(2u,-2v)
g(a,b,c)=max {ℓ(u-1,-1), ℓ(b-1,-(v+1))}
N(a,b,c) = N(a,b) - (b-u)v

Theorem 2

Let $c = \ell(u, -v) \in \overline{R}(a, b)$. If $s = \lceil \frac{b}{u} \rceil \leq \lfloor \frac{a}{v} \rfloor$, then s is the first representable multiple of c and

1. $g(a, b, c) = \max\{\ell((s-1)u - 1, -(s-2)v + 1), \ell(b - 1, -((s-1)v + 1))\}$
2. $N(a, b, c) = N(a, b) - v(s-1)b + vu\frac{s(s-1)}{2}$



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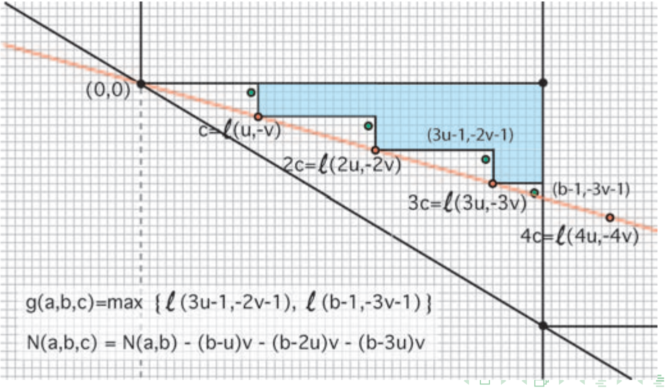
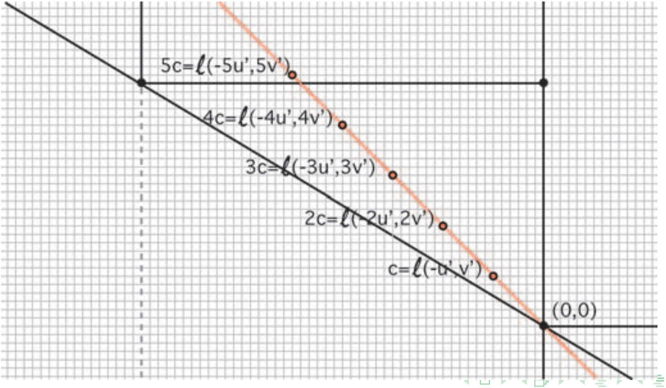


Theorem 3

Let $c = \ell(u, -v) \in \overline{R}(a, b)$ and $u' = b - u$, $v' = a - v$. If $s = \lceil \frac{a}{v'} \rceil \leq \lfloor \frac{b}{u'} \rfloor$, then s is the first representable multiple of c and

1. $g(a, b, c) = \max\{\ell(-(s-2)u' + 1, (s-1)v' - 1), \ell(-((s-1)u' + 1), a - 1)\}$
2. $N(a, b, c) = N(a, b) - u'(s-1)a + vu\frac{s(s-1)}{2}$



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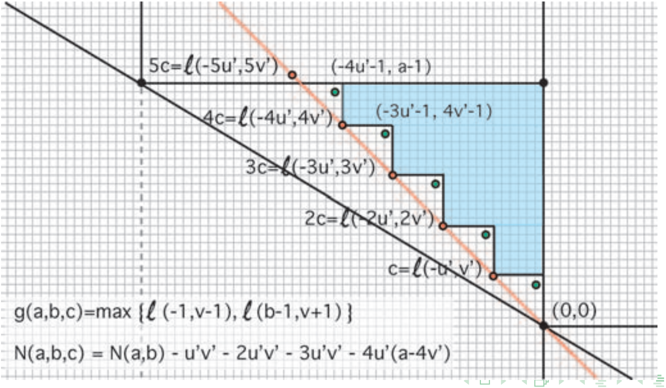
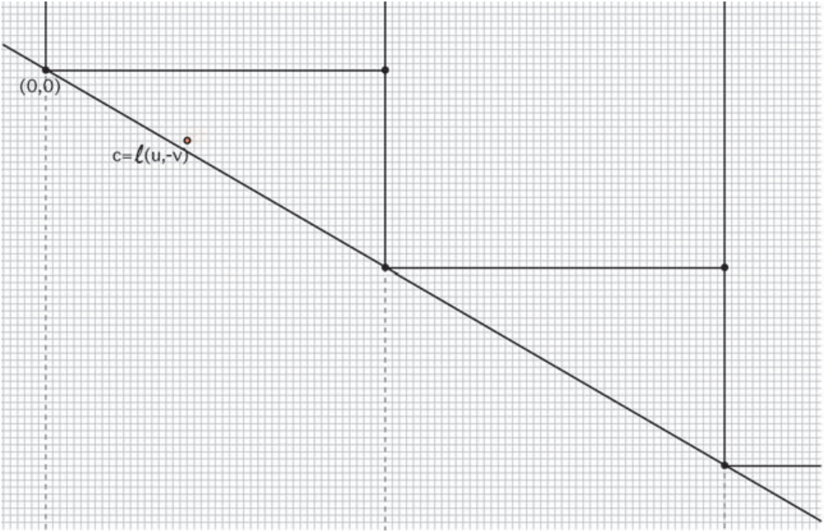


Not so easy...



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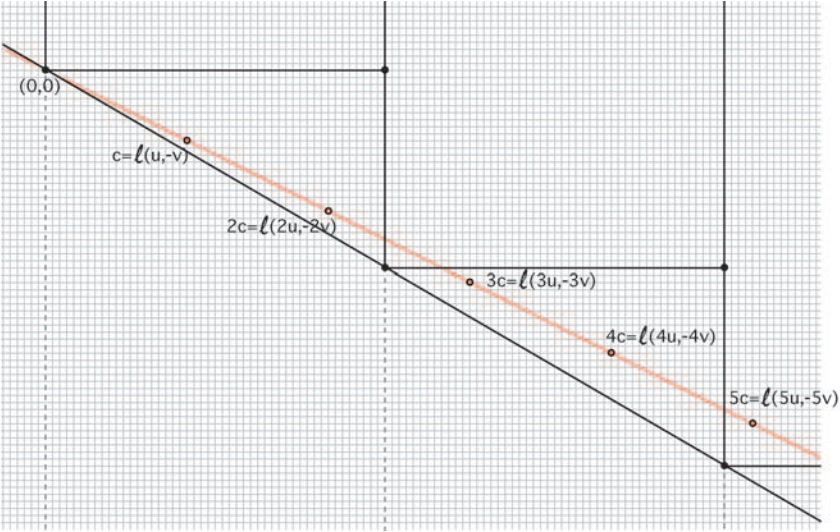

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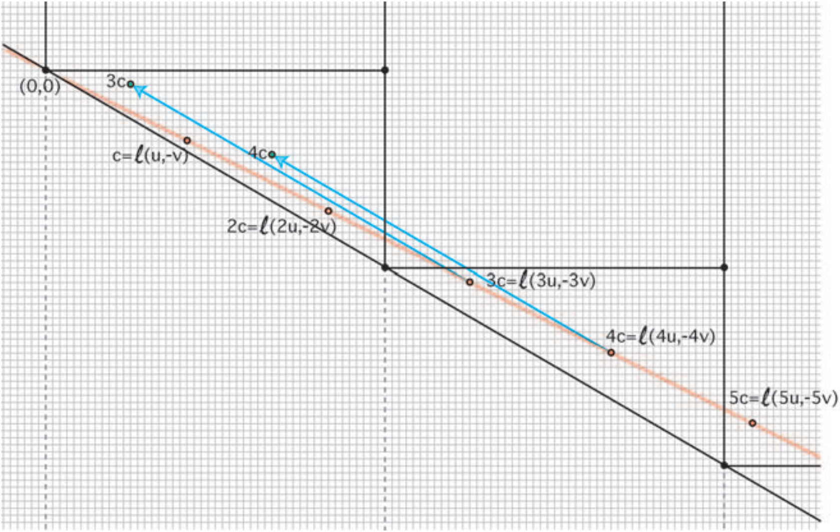

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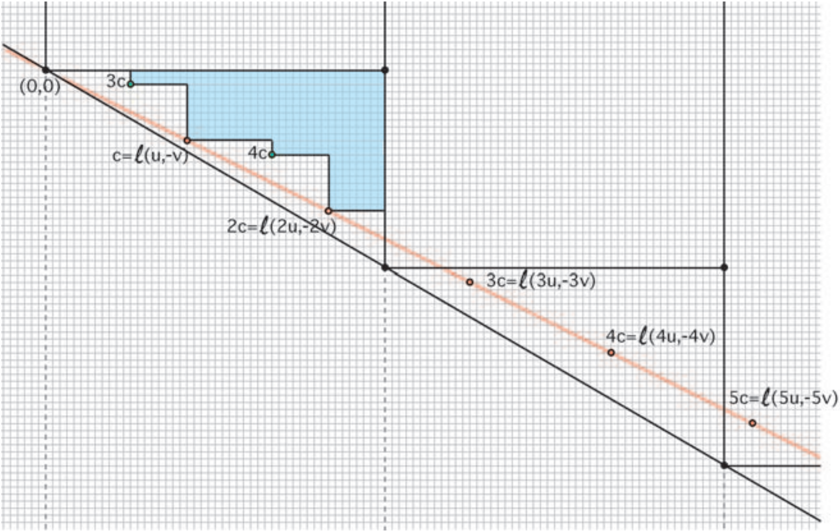

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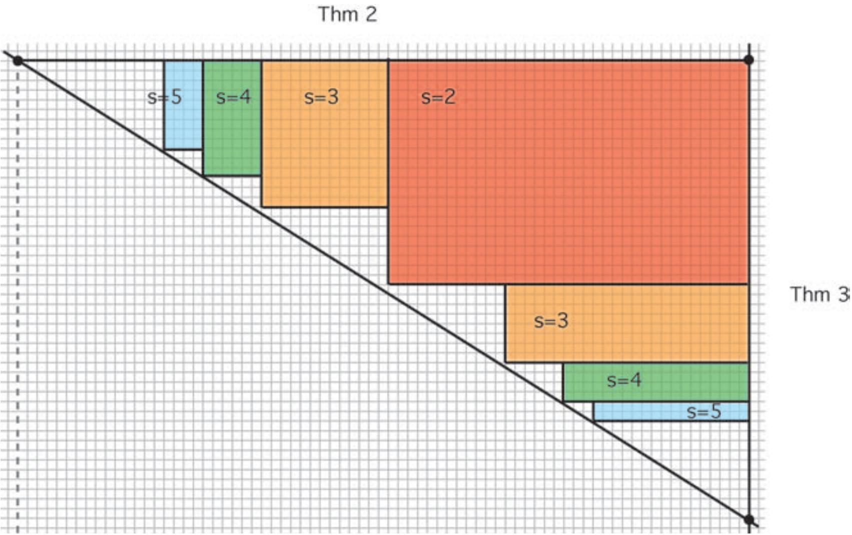

Geometric approach for the Frobenius problem:

- Compute $g(A)$
- Compute $N(A)$
- Represent the set of gaps

Work in progress

- Complete the triangle (in an easy way if possible!)
- Study the general case (remove the condition $(a, b) = 1$)
- Relate the plane representation for $\{a, b, c\}$ with the 3-dimensional lattice
- Apply similar techniques for $\{a, b, c, d\}$
- Generalize to arbitrary n

Thank you !!!