Some Properties on the Generalized Hierarchical Product of Graphs

Lali Barrière
Cristina Dalfó
Miquel Àngel Fiol
Margarida Mitjana

Universitat Politècnica de Catalunya

July 23, 2008

Outline

The hierarchical product

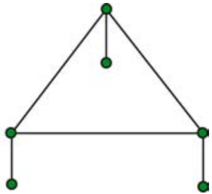
Definition [B, Comellas, Dalfó, Fiol, 2008]

For $i = 1, \ldots, N$, G_i graph rooted at 0, $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$

- vertices $x_N \ldots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in G_j then
  $x_N \ldots x_{j+1} x_j 0 \ldots 0 \sim x_N \ldots x_{j+1} y_j 0 \ldots 0$

Example

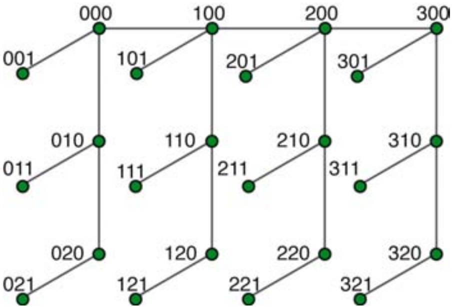
The hierarchical products $K_2 \sqcap K_3$ and $K_3 \sqcap K_2$



$G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is a spanning subgraph of $G_N \square \cdots \square G_2 \square G_1$

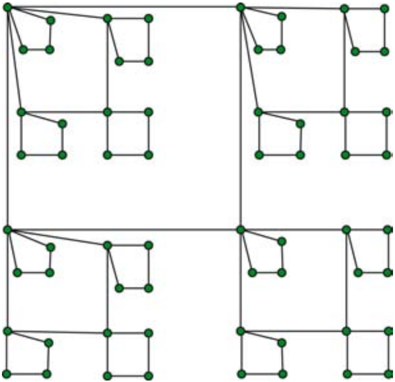
Example

The hierarchical product $P_4 \sqcap P_3 \sqcap P_2$



Example

The hierarchical power C_4^3



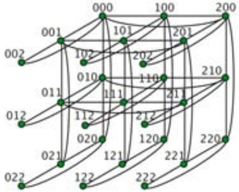
The generalized hierarchical product

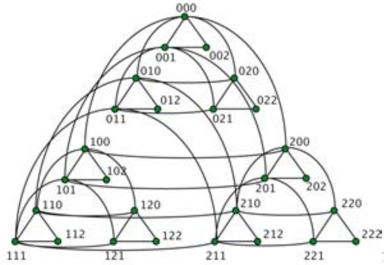
For $i = 1, \ldots N$, G_i graph, and for $i \leq N-1$, $\varnothing \neq U_i \subseteq V_i$,
$H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$

- vertices $x_N \ldots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in G_j and $u_i \in U_i$, $i = 1, \ldots, j-1$ then
  $x_N \ldots x_{j+1} x_j u_{j-1} \ldots u_1 \sim x_N \ldots x_{j+1} y_j u_{j-1} \ldots u_1$

Example

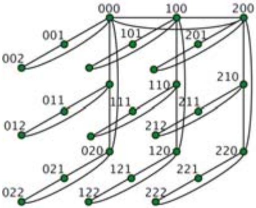
$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, *with* $U_1 = U_2 = \{0, 1\}$

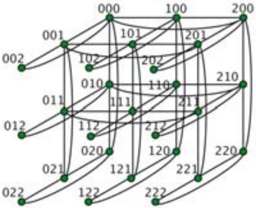


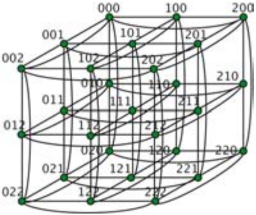


Paticular extreme cases

- (Standard) hierarchical product
  For all $1 \leq i \leq N-1$, $U_i = \{v_i\} \Rightarrow$
  $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$
  (assuming that v_i is labeled 0)
- Cartesian product
  For all $1 \leq i \leq N-1$, $U_i = V_i \Rightarrow$
  $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) = G_N \square \cdots \square G_2 \square G_1$







Hierarchy

$$H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$$

- $\mathbf{z} \in V_N \times \cdots \times V_{k+1} \Rightarrow H\langle \mathbf{z} x_k \ldots x_1 \rangle$ subgraph of H induced by the set $\{\mathbf{z} x_k \ldots x_1 \mid x_i \in V_i, 1 \leq i \leq k\}$
- $\mathbf{z} \in V_{k-1} \times \cdots \times V_1 \Rightarrow H\langle x_N \ldots x_k \mathbf{z} \rangle$ subgraph of H induced by the set $\{x_N \ldots x_k \mathbf{z} \mid x_i \in V_i, k \leq i \leq N\}$

Lemma

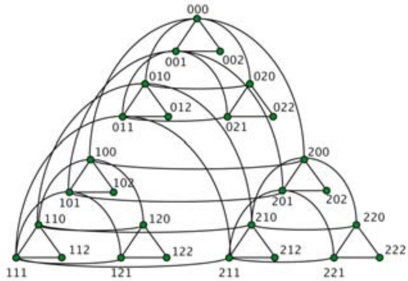
(a) $H\langle \mathbf{z} x_k \ldots x_1 \rangle = G_k \sqcap G_{k-1}(U_{k-1}) \sqcap \cdots \sqcap G_1(U_1)$

(b) $\mathbf{z} \in U_{k-1} \times \cdots \times U_1 \Rightarrow$
$H\langle x_N \ldots x_k \mathbf{z} \rangle = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_k(U_k)$

(c) $\mathbf{z} \notin U_{k-1} \times \cdots \times U_1 \Rightarrow H\langle x_N \ldots x_k \mathbf{z} \rangle = m\, K_1$, *where* $m = n_N \cdots n_k$

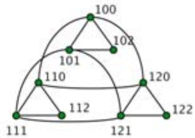
Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$



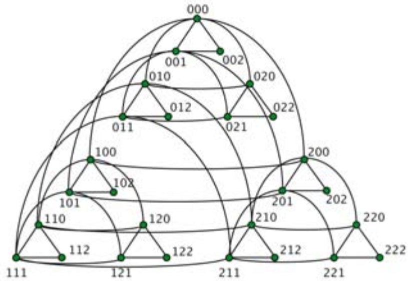
Example

$$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\}$$



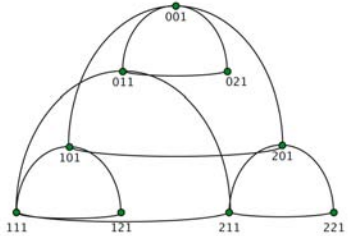
Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$



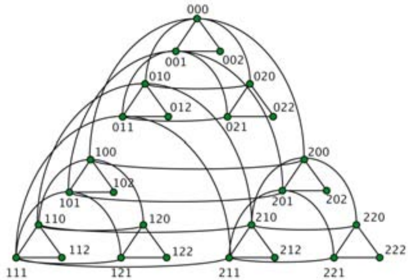
Example

$$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\}$$



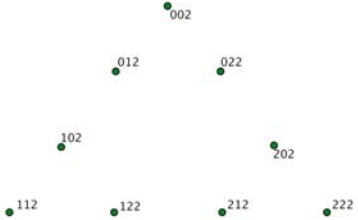
Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$



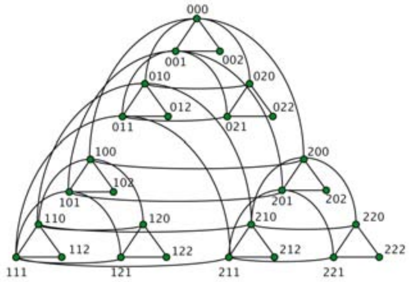
Example

$$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\}$$



Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$



Degree

- $v = x_N x_{N-1} \dots x_2 x_1 \in H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$

$$\begin{aligned}\deg_H(v) = \; & \deg_{G_1}(x_1) + \chi_{U_1}(x_1) \deg_{G_2}(x_2) + \cdots \\ & + \chi_{U_1}(x_1) \cdots \chi_{U_{N-1}}(x_{N-1}) \deg_{G_N}(x_N)\end{aligned}$$

- The minimum and maximum degree of H are

$$\begin{aligned}\delta_H = \; & \min\{\delta_{G_1(\overline{U}_1)}, \delta_{G_1(U_1)} + \delta_{G_2(\overline{U}_2)}, \dots, \\ & \delta_{G_1(U_1)} + \cdots + \delta_{G_{N-1}(U_{N-1})} + \delta_{G_N}\}\end{aligned}$$

$$\begin{aligned}\Delta_H = \; & \max\{\Delta_{G_1(\overline{U}_1)}, \Delta_{G_1(U_1)} + \Delta_{G_2(\overline{U}_2)}, \dots, \\ & \Delta_{G_1(U_1)} + \cdots + \Delta_{G_{N-1}(U_{N-1})} + \Delta_{G_N}\}\end{aligned}$$

For all $i = 1, 2, \dots, N$, $n_i = |V(G_i)|$

- For all $i = 1, 2, \dots, N$, G_i is δ_i-regular $\Rightarrow$ $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$ contains exactly
  - $n_N(n_{N-1} - |U_{N-1}|)$ vertices of degree δ_N
  - $n_N|U_{N-1}|(n_{N-2} - |U_{N-2}|)$ vertices of degree $\delta_N + \delta_{N-1}$

    $\vdots$
  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_2|(n_1 - |U_1|)$ vertices of degree $\delta_N + \cdots + \delta_2$
  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_1|$ vertices of degree $\delta_N + \cdots + \delta_1$

Associativity

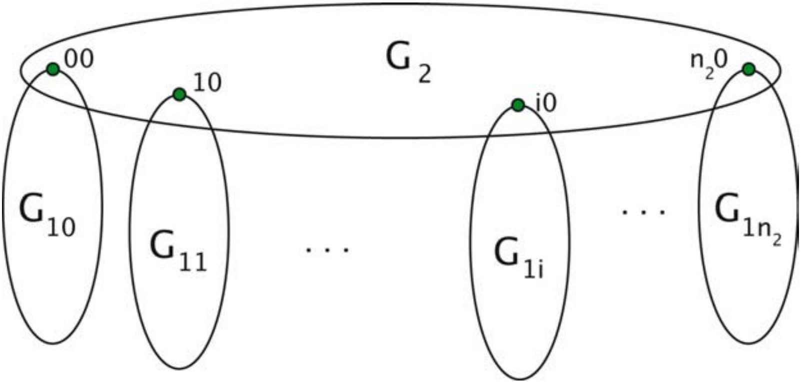
Proposition

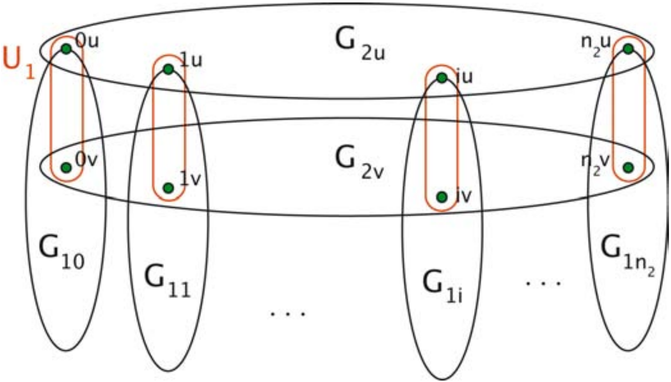
For $i = 1, 2, 3$, let G_i be a graph and, for $i = 1, 2$, $U_i \subseteq V_i$. The generalized hierarchical product satisfies

$$G_3 \sqcap G_2(U_2) \sqcap G_1(U_1) = G_3 \sqcap \left(G_2 \sqcap G_1(U_1)\right)(U_2 \times U_1)$$

$$= \left(G_3 \sqcap G_2(U_2)\right) \sqcap G_1(U_1)$$

Can be easily generalized to the case of N factors.


G 2
00
10
i0
n20
G10
G11
. . .
G1i
. . .
G1n2


U1
0u
1u
iu
n2u
G 2u
0v
1v
iv
n2v
G 2v
G10
G11
G1i
G1n2
. . .
. . .

Metric parameters

Definition (Distance through U)

$G = (V, E)$, $\varnothing \neq U \subset V$

- $p_{G(U)}(x, y)$ path from x to y *through* U, if intersects U
- $\text{dist}_{G(U)}(x, y) = \min \text{length}(p_{G(U)}(x, y))$

Remark $\text{dist}_{G(U)}(x, x) = 0 \Leftrightarrow x \in U$

- Mean distance through U: $d_{G(U)} = \frac{1}{n^2} \sum_{x,y \in V} \text{dist}_{G(U)}(x, y)$
- Eccentricity through U: $\text{ecc}_{G(U)}(x) = \max_{y \in V} \text{dist}_{G(U)}(x, y)$
- Radius through U: $r_{G(U)} = \min_{x \in V} \text{ecc}_{G(U)}(x)$
- Diameter through U: $D_{G(U)} = \max_{x \in V} \text{ecc}_{G(U)}(x)$

$$\mathbf{x} = (x_2, x_1),\ \mathbf{y} = (y_2, y_1) \in H = G_2 \sqcap G_1(U_1) \Rightarrow$$

$$\mathrm{dist}_H(\mathbf{x}, \mathbf{y}) = \mathrm{dist}_{G_2}(x_2, y_2) + \mathrm{dist}_{G_1(U_1)}(x_1, y_1)$$

Theorem

$H = G_2 \sqcap G_1(U_1),\ U_1 \subset V_1,\ n_2 = |V_2|$

(a) *Mean distance:*

$$d_H = d_{G_2} + \frac{1}{n_2}\left(d_{G_1} + (n_2 - 1)d_{G_1(U_1)}\right)$$

(b) *Eccentricity of* $\mathbf{x} = (x_2, x_1) \in V$*:*

$$\mathrm{ecc}_H(\mathbf{x}) = \mathrm{ecc}_{G_2}(x_2) + \mathrm{ecc}_{G_1(U_1)}(x_1)$$

(c) *Radius:*

$$r_H = r_{G_2} + r_{G_1(U_1)}$$

(d) *Diameter:*

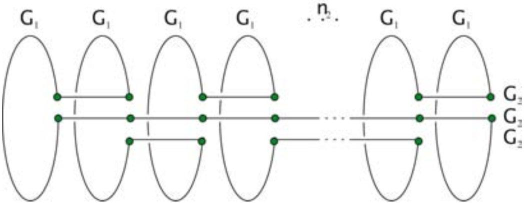
$$D_H = D_{G_2} + D_{G_1(U_1)}$$

Hamiltonian cycles: sufficient conditions

Proposition

If G_1, G_2 Hamiltonian and $G_1[U_1]$ contains two consecutive edges of a Hamiltonian cycle of G_1, then $H = G_2 \sqcap G_1(U_1)$ is Hamiltonian.

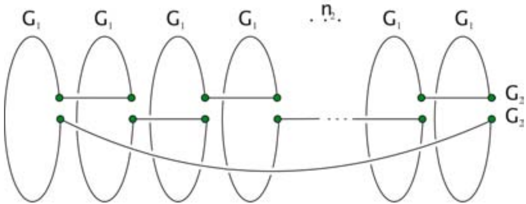
Sketch A Hamiltonian cycle in $G_2 \sqcap G_1(U_1)$ going through three copies of G_2 and n_2 copies of G_1.



Proposition

If G_1, G_2 Hamiltonian, $n_2 = |V_2|$ is even, and $G_1[U_1]$ contains one edge of a Hamiltonian cycle of G_1, then $H = G_2 \sqcap G_1(U_1)$ is Hamiltonian.

Sketch A Hamiltonian cycle in $G_2 \sqcap G_1(U_1)$ going through two copies of G_2 and n_2 copies of G_1 when n_2 is even.



Vertex-coloring

$\chi(G)$ chromatic number of G

- $\chi(G_2 \,\square\, G_1) = \max\{\chi(G_2), \chi(G_1)\}$ [Sabidussi, 1957]

Proposition

$\chi\big(G_2 \sqcap G_1(U_1)\big) = \max\{\chi(G_2), \chi(G_1)\}$

Proof.

$$G_1,\ G_2 < G_2 \sqcap G_1(U_1) < G_2 \square G_1$$

□

Edge-coloring

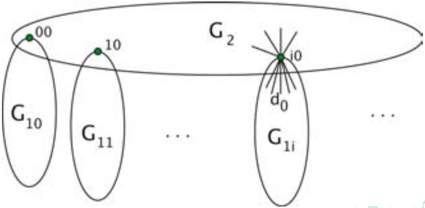
$\chi'(G)$ chromatic index of G

- $\Delta_G \leq \chi'(G) \leq \Delta_G + 1$ [Vizing, 1964]
  (G of class 1 if $\chi'(G) = \Delta_G$, G of class 2 if $\chi'(G) = \Delta_G + 1$)
- if one of the graphs G_1 or G_2 is of class 1 then $G_2 \square G_1$ is of class 1 [Mahmoodian, 1981]

Proposition

$\chi'(G_2 \sqcap G_1) = \max\{\Delta_{G_2} + d_0, \chi'(G_1)\}$

Proof.



Proposition

$$\max\{\Delta_{G_2} + \Delta_{G_1(U_1)}, \chi'(G_1)\} \leq \\ \leq \chi'(H) \leq \max\{\chi'(G_2) + \Delta_{G_1(U_1)}, \chi'(G_1)\}$$

Corollary

- *if* G_1 *is of class* 1 *and* $\exists u \in U_1$, $\deg_{G_1}(u) = \Delta_{G_1}$, *then* $\chi'(H) = \Delta_{G_2} + \Delta_{G_1}$
- *if* G_2 *is of class* 1 *then* $\chi'(H) = \max\{\Delta_{G_2} + \Delta_{G_1(U_1)}, \chi'(G_1)\}$

Vertex connectivity of $H = G_2 \sqcap G_1(U_1)$

Cartesian product: $H = G_2 \square G_1$, i.e., $U_1 = V_1$

$$\kappa(G_2 \square G_1) = \min\{\kappa_1|V_2|, \kappa_2|V_1|, \delta_1 + \delta_2\},$$

$\kappa_i = \kappa(G_i)$ and $\delta_i = \delta(G_i)$ [Špacapan 2007]

Connectivity relative to U

$G = (V, E)$ graph, $U \subsetneq V$

$$\kappa(U|\overline{U}) = \min\{|S| \, : \, S \subset V, \exists u \in G - S \text{ s.t. } \nexists \text{ path from } u \text{ to } U\}$$

Proposition

$U_1 \subsetneq V_1 \Rightarrow \kappa(G_2 \sqcap G_1(U_1)) \leq \min\{\kappa_1|V_2|, \kappa(U_1|\overline{U}_1), \delta_H\}$,
where $\delta_H = \min\{\delta_{G_1(\overline{U}_1)}, \delta_{G_1(U_1)} + \delta_{G_2}\}$.

See the poster on Algebraic properties of the generalized hierarchical product !!!

Thank you !!!