On the Spectra of Hypertrees

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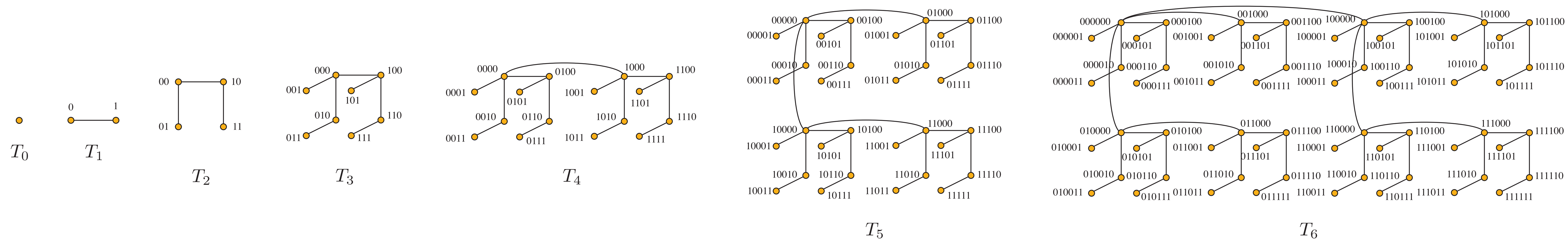


Definition

T_m is the rooted tree with $V = \mathbb{Z}_2^m$ and the adjacencies defined by: two vertices are adjacent iff they differ in exactly one position and their maximum common suffix either is empty or contains only 0s.

- The root of T_m is $\mathbf{0} = 00\ldots0$
- $T_0 = K_1$
- $T_m = K_2^m = K_2 \sqcap \overset{m}{\cdots} \sqcap K_2$, where the operator "$\sqcap$" indicates the hierarchical product.

Basic Properties

- T_m has order $n = 2^m$ and size $2^m - 1$.
- T_m is a spanning subtree of the hypercube Q_m.
- $T_m = T_{m-1} \sqcap K_2$
- $T_m^* = T_m - \vec{0} = \bigcup_{k=0}^{m-1} T_k$.
- $e = \{\mathbf{0}, 10 \overset{m-1}{\cdots} 0\} \Rightarrow T_m - e \cong T_m[V_0] \bigsqcup T_m[V_1]$ ($V_i = \{i\boldsymbol{w} | \boldsymbol{w} \in \mathbb{Z}_2^{m-1}\}$ and $T_m[V_i] \cong T_{m-1}$)

Degrees Sequence

T_m has 2 vertices of degree m and 2^{m-j} vertices of degree j:

- $\delta(\mathbf{0}) = \delta(10 \overset{m-1}{\cdots} 0) = m$
- $\delta(\boldsymbol{w}100 \overset{j-1}{\cdots} 0) = j$, for $\boldsymbol{w} \in \mathbb{Z}_2^{m-j}$ and $1 \le j \le m-1$.

Symmetry

Proposition 1

For every $m \ge 1$, the automorphism group of T_m is S_2.

Spectral Properties

The Adjacency Matrix of T_m

$$\boldsymbol{A}_0 = (0) \qquad \boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_3 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \cdots \qquad \boldsymbol{A}_m = \begin{pmatrix} \boldsymbol{A}_{m-1} & \boldsymbol{I} \\ \boldsymbol{I} & \mathbf{0} \end{pmatrix}$$

where the dimensions of each block are $2^{m-1} \times 2^{m-1}$

The Characteristic Polynomial of T_m

$$\phi_m(x) = \det(x\boldsymbol{I} - \boldsymbol{A}_m) = \det \begin{pmatrix} x\boldsymbol{I} - \boldsymbol{A}_{m-1} & -\boldsymbol{I} \\ -\boldsymbol{I} & x\boldsymbol{I} \end{pmatrix} = \det((x^2-1)\boldsymbol{I} - x\boldsymbol{A}_{m-1}) = \det\left(x\left((x - \tfrac{1}{x})\boldsymbol{I} - \boldsymbol{A}_{m-1}\right)\right) = x^{\frac{n}{2}} \phi_{m-1}(x - \tfrac{1}{x})$$

Eigenvalues

$\phi_m(x) = x^{\frac{n}{2}} \phi_{m-1}(x - \frac{1}{x}) \Rightarrow$

If $\lambda_{\boldsymbol{i}} \in \operatorname{ev} T_{m-1}$ and $\lambda_{0\boldsymbol{i}}$, $\lambda_{1\boldsymbol{i}}$ are the solutions of $x^2 - \lambda_{\boldsymbol{i}} x - 1 = 0$, then $\lambda_{0\boldsymbol{i}}, \lambda_{1\boldsymbol{i}} \in \operatorname{ev} T_m$.

$$\left.\begin{array}{l} f_0(\lambda) := \frac{1}{2}(\lambda - \sqrt{\lambda^2+4}) \\ f_1(\lambda) := \frac{1}{2}(\lambda + \sqrt{\lambda^2+4}) \end{array}\right\} \Rightarrow \lambda_{0\boldsymbol{i}} = f_0(\lambda_{\boldsymbol{i}}) \text{ and } \lambda_{1\boldsymbol{i}} = f_1(\lambda_{\boldsymbol{i}})$$

$$\left.\begin{array}{l} \lambda_\varnothing := 0 \\ \operatorname{ev} T_m = \{\lambda_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^m\} \\ \boldsymbol{i} = i_{m-1} i_{m-2} \ldots i_0 \end{array}\right\} \Rightarrow \lambda_{\boldsymbol{i}} = (f_{i_{m-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0})(0)$$

Properties

- Every eigenvalue of T_{m-1} gives rise to two eigenvalues of T_m.
- All the 2^m eigenvalues are different: $\operatorname{ev} T_m = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$
- The natural order in $\mathbb{Z}_2^m$ gives a natural order in $\operatorname{ev} T_m$.
- For every $\boldsymbol{i} \in \mathbb{Z}_2^m$, $\lambda_{\boldsymbol{i}} = -\lambda_{\overline{\boldsymbol{\imath}}}$, where $\overline{\boldsymbol{\imath}} (= n - i)$ denotes the ones'complement of $\boldsymbol{i}$.

Asymptotic Behavior

Proposition 2

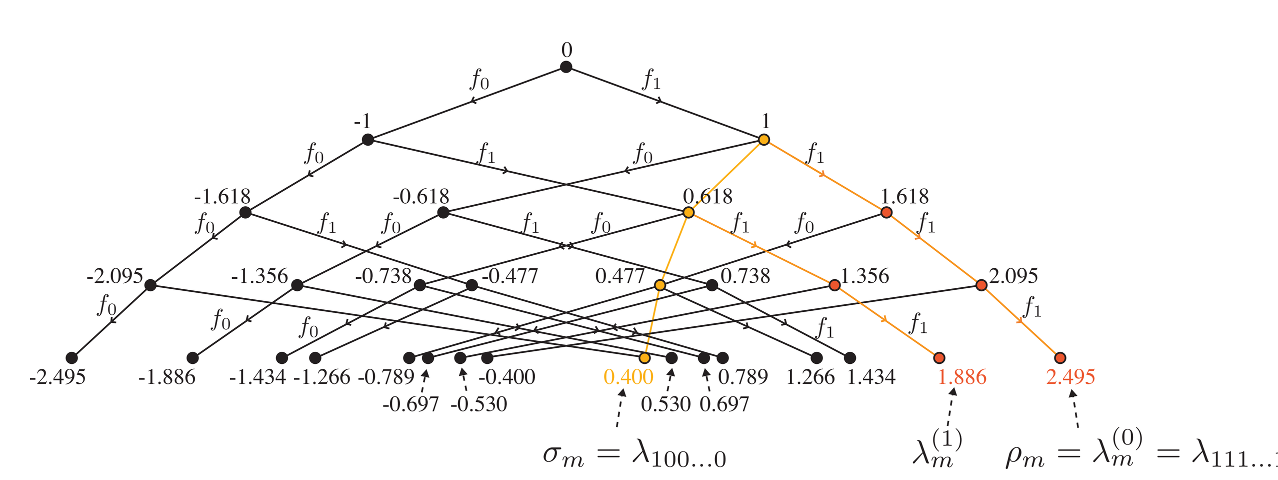
The asymptotic behaviors of the maximum eigenvalue (spectral radius), $\rho_m = \max_{0 \le i \le n-1}\{|\lambda_i|\} = \lambda_{111\ldots1}$, and the minimum positive eigenvalue, $\sigma_m = \min_{0 \le i \le n-1}\{|\lambda_i|\} = \lambda_{100\ldots0}$, of the hypertree T_m are

$$\rho_m \sim \sqrt{2m}, \quad \sigma_m \sim 1/\sqrt{2m}$$

Proposition 3

For every fixed $r \ge 0$, let $\{\lambda_m^{(r)} = \lambda_{111\ldots1\overline{\boldsymbol{r}}}\}_{m \ge k}$, that is, $\lambda_m^{(r)} = \gamma_m$ denotes the $(r+1)$-th largest eigenvalue of T_m. Then, the asymptotic behavior of $\lambda_m^{(r)}$ is:

$$\lambda_m^{(r)} \sim \sqrt{2m}.$$



Distribution of All the Eigenvalues of Hypertrees

Definition (Order in $\bigcup_m \operatorname{ev} T_m$) Let $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_2^*$ and let $\boldsymbol{w}$ be their maximum common prefix. We say that $\boldsymbol{i} <_T \boldsymbol{j}$ iff
$$\begin{cases} \boldsymbol{i} = \boldsymbol{w}0\boldsymbol{i}_1 \text{ and } \boldsymbol{j} = \boldsymbol{w}1\boldsymbol{j}_1; \text{ or} \\ \boldsymbol{i} = \boldsymbol{w} \text{ and } \boldsymbol{j} = \boldsymbol{w}1\boldsymbol{j}_1; \text{ or} \\ \boldsymbol{i} = \boldsymbol{w}0\boldsymbol{i}_1 \text{ and } \boldsymbol{j} = \boldsymbol{w} \end{cases}$$

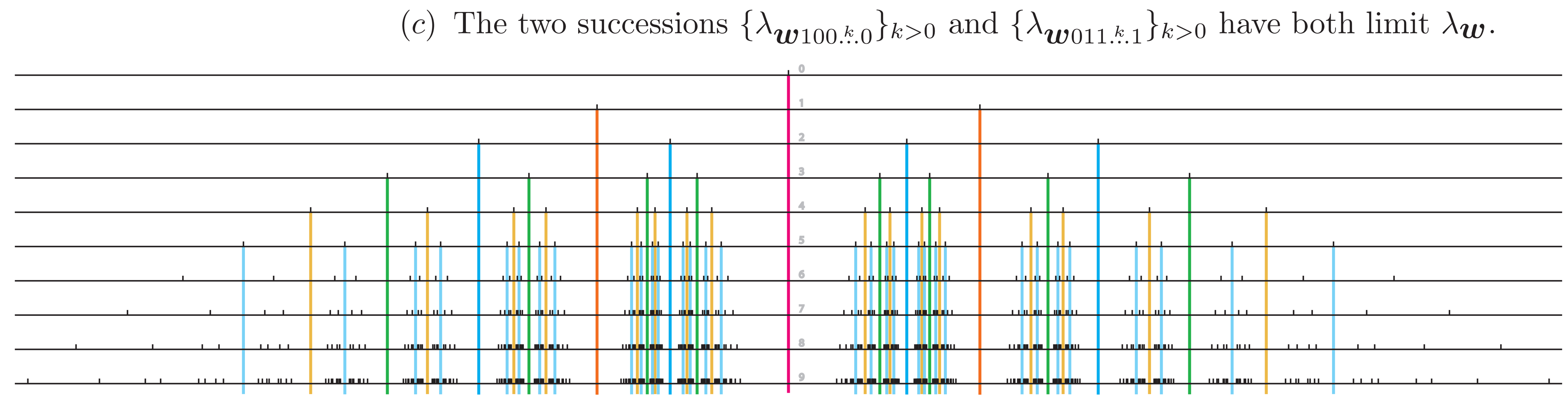
Theorem 4

The set of all the eigenvalues, $\bigcup_m \operatorname{ev} T_m = \{\lambda_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathbb{Z}_2^*\}$, satisfies:

(*a*) For every $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_2^*$, $\boldsymbol{i} <_T \boldsymbol{j}$ if and only if $\lambda_{\boldsymbol{i}} < \lambda_{\boldsymbol{j}}$.

(*b*) The interval determined by two consecutive eigenvalues of dimension m, contains exactly 2^k consecutive eigenvalues of T_{m+k}, for $k \ge 1$.

(*c*) The two successions $\{\lambda_{\boldsymbol{w}100\overset{k}{\ldots}0}\}_{k>0}$ and $\{\lambda_{\boldsymbol{w}011\overset{k}{\ldots}1}\}_{k>0}$ have both limit $\lambda_{\boldsymbol{w}}$.

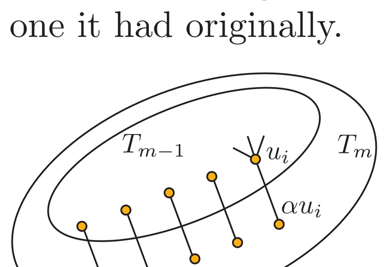


Eigenvectors

For any digraph, the components of its eigenvalues can be seen as charges on each vertex. The charge of a vertex $i \in V$ is the corresponding entry v_i of $\boldsymbol{v}$, and the equation $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$ means that

$$\sum_{i \to j} v_j = \lambda v_i \quad \text{for every} \quad i \in V.$$

That is, each vertex "absorbs" the charges of its out-neighbors to get a final charge λ times the one it had originally.

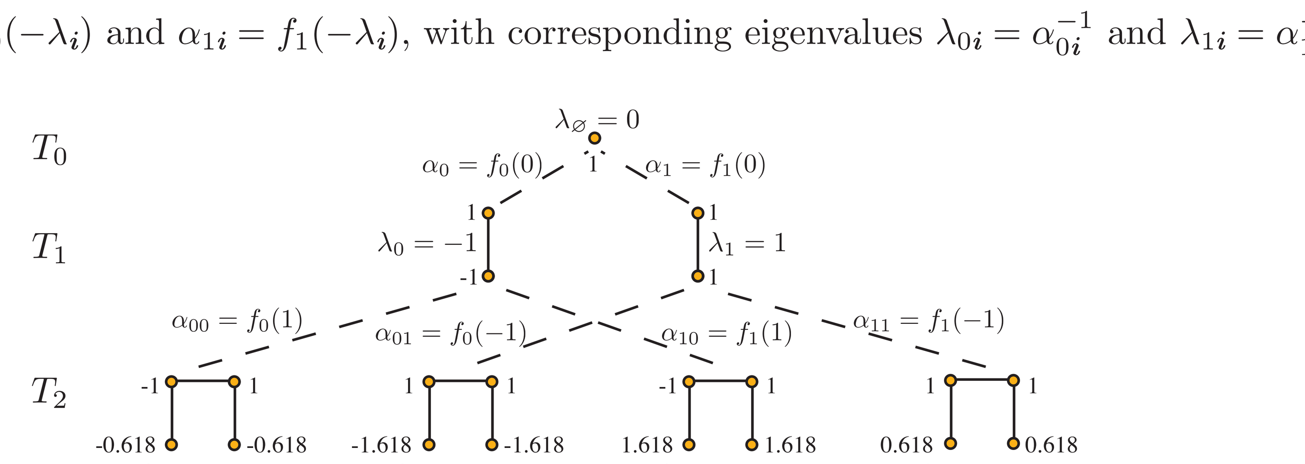


Proposition 5

Every $\lambda_{\boldsymbol{i}}$-eigenvector $\boldsymbol{u}_{\boldsymbol{i}}$ of the hypertree T_{m-1} gives rise to the following eigenvectors of T_m:

$$\boldsymbol{u}_{0\boldsymbol{i}} = (\boldsymbol{u}_{\boldsymbol{i}}, \alpha_{0\boldsymbol{i}}\boldsymbol{u}_{\boldsymbol{i}})^\top, \quad \boldsymbol{u}_{1\boldsymbol{i}} = (\boldsymbol{u}_{\boldsymbol{i}}, \alpha_{1\boldsymbol{i}}\boldsymbol{u}_{\boldsymbol{i}})^\top,$$

where $\alpha_{0\boldsymbol{i}} = f_0(-\lambda_{\boldsymbol{i}})$ and $\alpha_{1\boldsymbol{i}} = f_1(-\lambda_{\boldsymbol{i}})$, with corresponding eigenvalues $\lambda_{0\boldsymbol{i}} = \alpha_{0\boldsymbol{i}}^{-1}$ and $\lambda_{1\boldsymbol{i}} = \alpha_{1\boldsymbol{i}}^{-1}$.



References

[1] L. Barrière, F. Comellas, C. Dalfó, and M.A. Fiol, The hierarchical product of graphs, Discrete Appl. Math., submitted. (Available at http://hdl.handler.net/2117/672.)
[2] L. Barrière, F. Comellas, C. Dalfó, and M.A. Fiol, On the Spectra of Hypertrees, Linear Algebra Appl., submitted. (Available at http://hdl.handle.net/2117/891.)
[3] N. Biggs, Algebraic Graph Theory, Cambridge Univ. Press, Cambridge, 1974, second edition, 1993.
[4] F.R.K. Chung, Diameters and eigenvalues, J. Amer. Math. Soc. 2 (1989) 187–196.
[5] M.A. Fiol and M. Mitjana, The spectra of some families of digraphs, Linear Algebra Appl. 423 (2007) 109–118.
[6] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, New York, 1993.
[7] A. Mowshowitz, The group of a graph whose adjacency matrix has all distinct eigenvalues, in F. Harary ed., Proof Techniques in Graph Theory, Academic Press, New York, 1969, pp. 109–110.
[8] J. R. Silvester, Determinants of block matrices, Maths Gazette 84 (2000), 460–467.

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