

The Hierarchical Product of Graphs

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2 The hierarchical product

Definition and basic properties

Vertex hierarchy

Metric parameters

3 Algebraic properties

Spectral properties of $G \square K_2^m$

The spectrum of the binary hypertree $T_m = K_2^m$

The spectrum of a generic two-term product $\tilde{G}_2 \square G_1$

4 Generalization of the hierarchical product

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Motivation

Complex networks: randomness, heterogeneity, modularity

- M.E.J. Newman. The structure and function of complex networks. SIAM Rev. 45 (2003) 167–256.

Hierarchical networks: degree distribution, modularity

- S. Jung, S. Kim, B. Kahng. Geometric fractal growth model for scale-free networks. Phys. Rev. E 65 (2002) 056101.
- E. Ravasz, A.-L. Barabási, Hierarchical organization in complex networks, Phys. Rev. E 67 (2003) 026112.
- E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Oltvai, A.-L. Barabási, Hierarchical organization of modularity in metabolic networks, Science 297 (2002) 1551–1555.

Our work

- Deterministic graphs
- Algebraic methods

Our work

- Deterministic graphs
- Algebraic methods
- Far from "real networks"

but a beautiful mathematical object !!!

Previous work

- N. Biggs. Algebraic Graph Theory. Cambridge UP, Cambridge, 1974.
- D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs. Theory and Applications, Academic Press, New York, 1980.
- M.A. Fiol, M. Mitjana. The local spectra of regular line graphs. Discrete Math., submitted.
- C. D. Godsil. Algebraic Combinatorics. Chapman and Hall, New York, 1993.
- A.J. Schwenk, Computing the characteristic polynomial of a graph, Lect. Notes Math. 406 (1974) 153–172.
- J. R. Silvester, Determinants of block matrices, Maths Gazette 84 (2000) 460–467.

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Definition

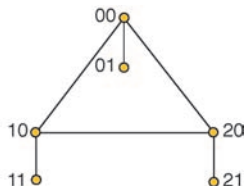
For $i = 1, \dots, N$, G_i graph rooted at 0

$$H = G_N \square \dots \square G_2 \square G_1$$

- vertices $x_N \dots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in G_j then
 $x_N \dots x_{j+1} x_j 0 \dots 0 \sim x_N \dots x_{j+1} y_j 0 \dots 0$

Example

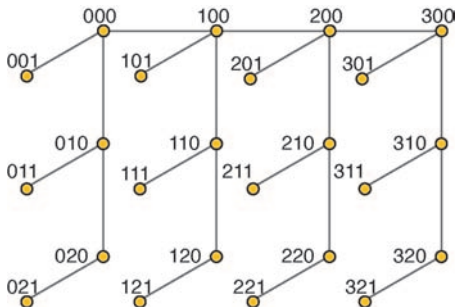
The hierarchical products $K_2 \square K_3$ and $K_3 \square K_2$



$G_N \square \cdots \square G_2 \square G_1$ is a spanning subgraph of $G_N \square \cdots \square G_2 \square G_1$

Example

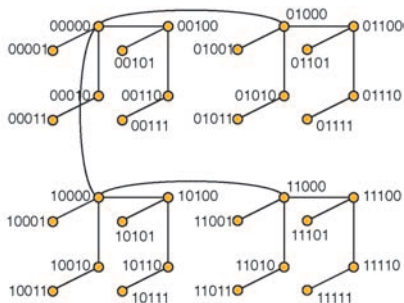
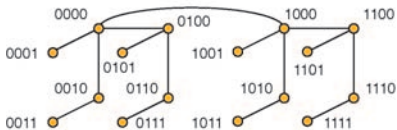
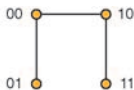
The hierarchical product $P_4 \square P_3 \square P_2$



$G^N = G \square \dots \square G$ is the hierarchical N -power of G

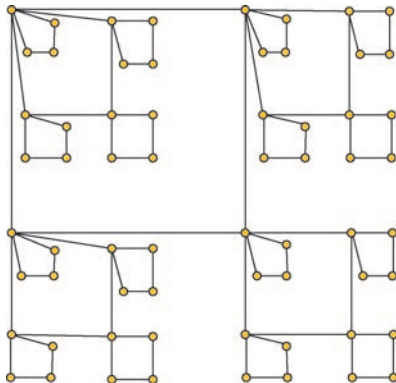
Example

The hierarchical powers K_2^2 , K_2^4 and K_2^5



Example

The hierarchical power C_4^3



Order and size

$$n_i = |V_i| \text{ and } m_i = |E_i|$$

$$H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$$

- $n_H = n_N \cdots n_3 n_2 n_1$
- $m_H = \sum_{k=1}^N m_k n_{k+1} \cdots n_N$

Properties of \sqcap

- Associativity. $G_3 \sqcap G_2 \sqcap G_1 = G_3 \sqcap (G_2 \sqcap G_1) = (G_3 \sqcap G_2) \sqcap G_1$
- Right-distributivity. $(G_3 \cup G_2) \sqcap G_1 = (G_3 \sqcap G_1) \cup (G_2 \sqcap G_1)$
- Left-semi-distributivity. $G_3 \sqcap (G_2 \cup G_1) = (G_3 \sqcap G_2) \cup n_3 G_1$,
where $n_3 G_1 = \overline{K}_{n_3} \sqcap G_1$ is n_3 copies of G_1
- $G \sqcap K_1 = K_1 \sqcap G = G$

(\mathcal{G}, \sqcap) is a monoid

Degrees

- If $\delta_i = \delta_{G_i}(0)$, then

- $\delta_G(\mathbf{0}) = \sum_{i=1}^N \delta_i$

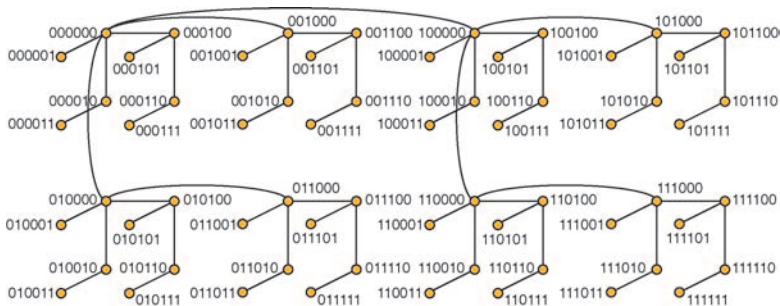
- $\mathbf{x} = x_N x_{N-1} \dots x_k 0 0 \dots 0, x_k \neq 0 \Rightarrow$

$$\delta_H(\mathbf{x}) = \sum_{i=1}^{k-1} \delta_i + \delta_{G_k}(x_k)$$

- If G is δ -regular, the degrees of the vertices of G^N follow an exponential distribution, $P(k) = \gamma^{-k}$, for some constant γ
For $k = 1, \dots, N - 1$, G^N contains $(n - 1)n^{N-k}$ vertices with degree $k\delta$ and n vertices with degree $N\delta$

Example

$T_m = K_2^m$ has 2^{m-k} vertices of degree $k = 1, \dots, m-1$, and two vertices of degree m



Modularity

$H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$, \mathbf{z} an appropriate string

$$H\langle \mathbf{z}x_k \dots x_1 \rangle = H[\{\mathbf{z}x_k \dots x_1 \mid x_i \in V_i, 1 \leq i \leq k\}]$$

$$H\langle x_N \dots x_k \mathbf{z} \rangle = H[\{x_N \dots x_k \mathbf{z} \mid x_i \in V_i, k \leq i \leq N\}]$$

Lemma

- $H\langle \mathbf{z}x_k \dots x_1 \rangle = G_k \sqcap \cdots \sqcap G_1$, for any fixed \mathbf{z}
- $H\langle x_N \dots x_k \mathbf{0} \rangle = G_N \sqcap \cdots \sqcap G_k$
- $H\langle x_N \dots x_k \mathbf{z} \rangle = (n_N \cdots n_k)K_1$, for any fixed $\mathbf{z} \neq \mathbf{0}$

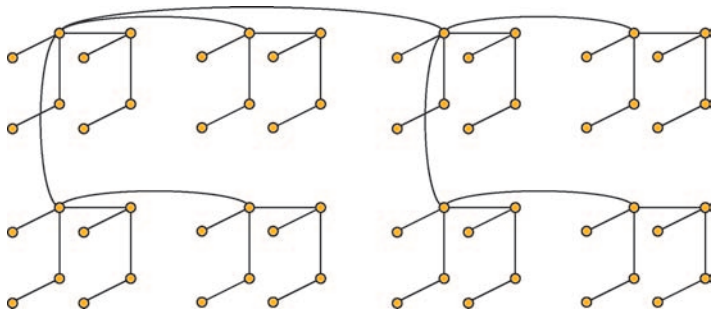
$$H^* = H - \mathbf{0}$$

Lemma

- $(G_N \sqcap \cdots \sqcap G_2 \sqcap G_1)^* = \bigcup_{k=1}^N (G_k^* \sqcap G_{k-1} \sqcap \cdots \sqcap G_1)$
- $(K_2^N)^* = \bigcup_{k=0}^{N-1} K_2^k$
- $K_2^N - \{\{\mathbf{0}, \mathbf{10}\}\} = K_2^{N-1} \cup K_2^{N-1}$

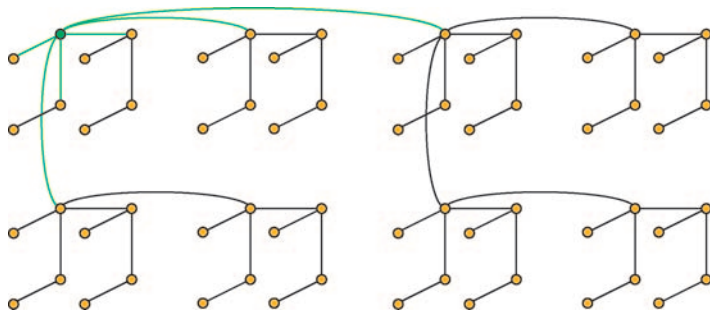
Example

Modularity and symmetry of $T_m = K_2^m$



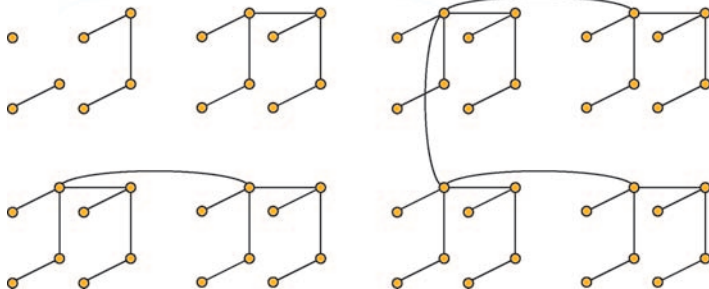
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Modularity and symmetry of $T_m = K_2^m$



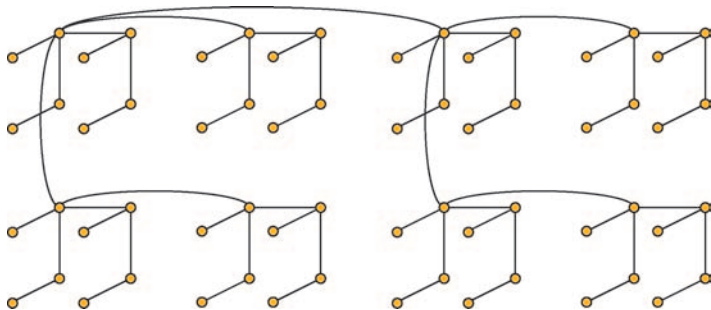
Example

Modularity and symmetry of $T_m = K_2^m$



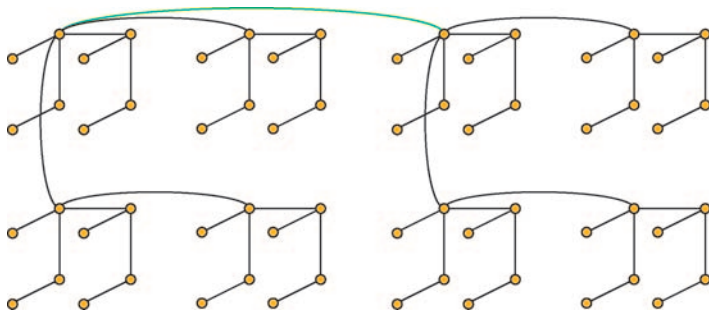
Example

Modularity and symmetry of $T_m = K_2^m$



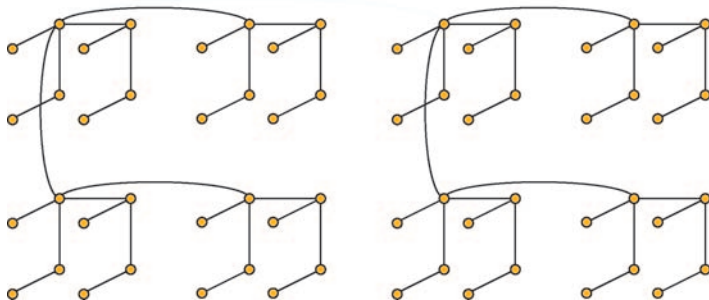
Example

Modularity and symmetry of $T_m = K_2^m$



Example

Modularity and symmetry of $T_m = K_2^m$



Eccentricity, radius and diameter

$$H = G_N \square \cdots \square G_2 \square G_1$$

$$\varepsilon_i = \text{ecc}_{G_i}(0), \quad r_{G_N} = r_N \text{ and } D_{G_N} = D_N$$

ρ_i shortest path routing of G_i , $i = 1, \dots, N$

Proposition

- $\{\rho_i\}_{i=1\dots N}$ induce a shortest path routing ρ in H
- The eccentricity, radius and diameter of H are

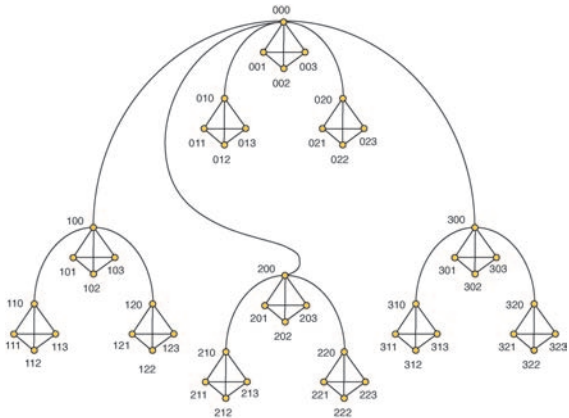
$$\text{ecc}_H(\mathbf{0}) = \sum_{i=1}^N \varepsilon_i, \quad r_H = r_N + \sum_{i=1}^{N-1} \varepsilon_i, \quad D_H = D_N + 2 \sum_{i=1}^{N-1} \varepsilon_i$$

The Hierarchical Product of Graphs

The hierarchical product

Metric parameters

Proof.

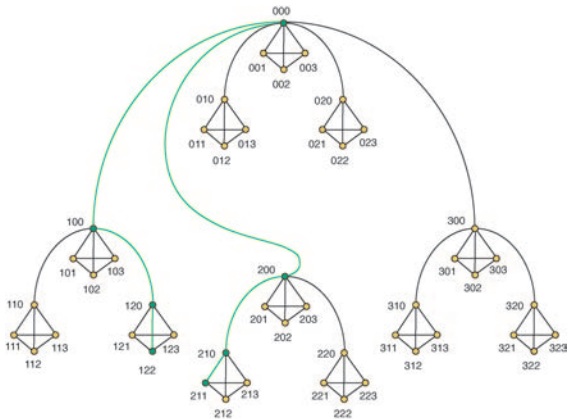


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Proof.



Mean distance

G graph of order n

Mean distance. $d_G = \frac{1}{n(n-1)} \sum_{v \neq w \in V} \text{dist}_G(v, w)$

Local mean distance. $d_G^0 = \frac{1}{n} \sum_{v \in V} \text{dist}_G(0, v)$

Proposition

$$H = G_2 \square G_1 \Rightarrow \begin{cases} d_H^{00} = d_1^0 + d_2^0 \\ d_H = \frac{1}{n-1} [(n_1 - 1)d_1 + n_1(n_2 - 1)(d_2 + 2d_1^0)] \end{cases}$$

Mean distance

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Proof.

Just compute!



Corollary

$$H = G^N, d = d_G, d^0 = d_G^0$$

- $\text{ecc}_N(\mathbf{0}) = N\varepsilon, d_N^{\mathbf{0}} = Nd^0$
- $r_N = r + (N - 1)\varepsilon, D_N = D + 2(N - 1)\varepsilon$
- $d_N = d + 2 \left(\frac{(N-1)n^{N+1}}{n^N - 1} - \frac{1}{n-1} \right) d^0$

$$\text{Asymptotically, } d_N \underset{N}{\sim} d + 2d^0 \left(N - \frac{n}{n-1} \right) \underset{n}{\sim} d + 2Nd^0$$

Example

$$G = K_2 \Rightarrow \text{ecc} = r = D = 1, d^0 = 1/2 \text{ and } d = 1$$

The metric parameters of $T_m = K_2^m$ are

- $\text{ecc}_m(\mathbf{0}) = m, d_m^{\mathbf{0}} = m/2$
- $r_m = m, D_m = 2m - 1$
- $d_m = \frac{m2^m}{2^m - 1} - 1 \sim m - 1$

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Background

Kronecker product $\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$

If \mathbf{A} and \mathbf{B} are square, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are permutation similar

Background

Kronecker product $\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$

If \mathbf{A} and \mathbf{B} are square, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are permutation similar

Lemma

$H = G_2 \square G_1 \Rightarrow$

$$\mathbf{A}_H = \mathbf{A}_2 \otimes \mathbf{D}_1 + \mathbf{I}_2 \otimes \mathbf{A}_1 \cong \mathbf{D}_1 \otimes \mathbf{A}_2 + \mathbf{A}_1 \otimes \mathbf{I}_2$$

where $D_1 = \text{diag}(1, 0, \dots, 0)$

Background

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where $D_1 = \text{diag}(1, 0, \dots, 0)$

Example

$H = G \square K_n$, G of order $N \Rightarrow$

$$\mathbf{A}_H = \mathbf{D}_1 \otimes \mathbf{A}_G + \mathbf{A}_{K_n} \otimes \mathbf{I}_N = \begin{pmatrix} \mathbf{A}_G & \mathbf{I}_N & \cdots & \mathbf{I}_N \\ \mathbf{I}_N & \mathbf{0} & \cdots & \mathbf{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}_N & \mathbf{I}_N & \cdots & \mathbf{0} \end{pmatrix}$$

Theorem (Sylvester, 2000)

R commutative subring of $F^{n \times n}$, the set of all $n \times n$ matrices over a field F (or a commutative ring), and $\mathbf{M} \in R^{m \times m}$. Then,

$$\det_F \mathbf{M} = \det_F(\det_R \mathbf{M})$$

Corollary (Sylvester, 2000)

$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}$ where $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ commute with each other.

Then,

$$\det \mathbf{M} = \det(\mathbf{AD} - \mathbf{BC})$$

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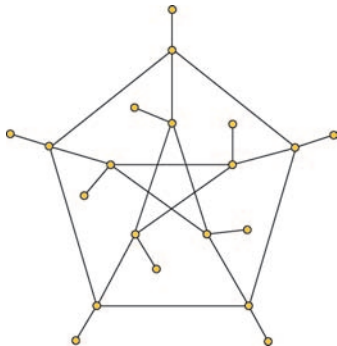
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$G \square K_2$

Example

The Petersen graph, hierarchically multiplied by K_2



$G \square K_2$

G graph of order n ,

\mathbf{A} adjacency matrix of G and

ϕ_G characteristic polynomial of G

- The adjacency matrix of $H = G \square K_2$ is

$$\mathbf{A}_H = \begin{pmatrix} \mathbf{A} & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix}$$

- The characteristic polynomial of H is

$$\begin{aligned} \phi_H(x) &= \det(x\mathbf{I}_{2n} - \mathbf{A}_H) = \det \begin{pmatrix} x\mathbf{I}_n - \mathbf{A} & -\mathbf{I}_n \\ -\mathbf{I}_n & x\mathbf{I}_n \end{pmatrix} = \\ &= \det((x^2 - 1)\mathbf{I}_n - x\mathbf{A}) = x^n \phi_G(x - \frac{1}{x}) \end{aligned}$$

$$\phi_H(x) = x^n \phi_G(x - \frac{1}{x})$$

Proposition

$H = G \square K_2$ and $\text{sp}G = \{\lambda_0^{m_0} < \lambda_1^{m_1} < \dots < \lambda_d^{m_d}\} \Rightarrow$

$$\text{sp}H = \{\lambda_{00}^{m_0} < \lambda_{01}^{m_1} < \dots < \lambda_{0d}^{m_d} < \lambda_{10}^{m_0} < \lambda_{11}^{m_1} < \dots < \lambda_{1d}^{m_d}\}$$

where $\lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}$, $\lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2}$

$$\phi_H(x) = x^n \phi_G(x - \frac{1}{x})$$

Proposition

$$H = G \square K_2 \text{ and } \text{sp}G = \{\lambda_0^{m_0} < \lambda_1^{m_1} < \dots < \lambda_d^{m_d}\} \Rightarrow$$

$$\text{sp}H = \{\lambda_{00}^{m_0} < \lambda_{01}^{m_1} < \dots < \lambda_{0d}^{m_d} < \lambda_{10}^{m_0} < \lambda_{11}^{m_1} < \dots < \lambda_{1d}^{m_d}\}$$

$$\text{where } \lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}, \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2}$$

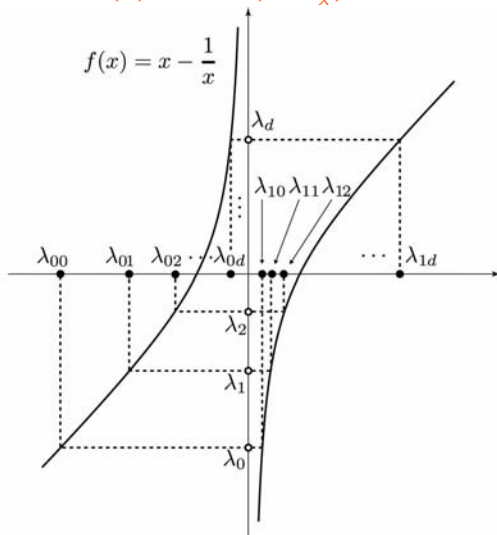
Proof.

$$\lambda \in \text{sp}H \Leftrightarrow \phi_H(\lambda) = \lambda^n \phi_G(\lambda - \frac{1}{\lambda}) = 0 \Leftrightarrow \lambda - \frac{1}{\lambda} \in \text{sp}G$$

$$\lambda_i \in \text{sp}G \Rightarrow \lambda^2 - \lambda_i \lambda - 1 = 0$$



$$\phi_H(x) = x^n \phi_G(x - \frac{1}{x})$$



$$H_m = G \square K_2^m$$

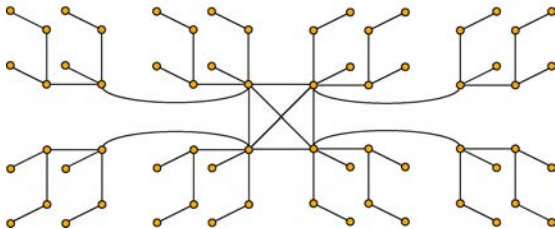
$H_m = H_{m-1} \square K_2$, $m \geq 1$. The adjacency matrix of H_m is

$$\mathbf{A}_m = \begin{pmatrix} \mathbf{A}_{m-1} & \mathbf{I}_{m-1} \\ \mathbf{I}_{m-1} & \mathbf{0} \end{pmatrix}$$

where \mathbf{I}_m denotes the identity matrix of size $n2^m$ (the same as \mathbf{A}_m)

$H_0 = G$, $\mathbf{A}_0 = \mathbf{A}$ the adjacency matrix of G

Example



Let $\{p_i, q_i\}_{i \geq 0}$ be the family of polynomials satisfying the recurrence equations

$$p_i = p_{i-1}^2 - q_{i-1}^2$$

$$q_i = p_{i-1}q_{i-1}$$

with initial conditions

$$p_0 = x \text{ and } q_0 = 1$$

Proposition

For every $m \geq 0$, the characteristic polynomial of $H_m = G \square K_2^m$ is

$$\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right)$$

Lemma

If p and q are arbitrary polynomials, then

$$\det \begin{pmatrix} p\mathbf{I}_n - q\mathbf{A} & -q\mathbf{I}_n \\ -q\mathbf{I}_n & p\mathbf{I}_n \end{pmatrix} = \det((p^2 - q^2)\mathbf{I}_n - pq\mathbf{A})$$

Proof of $\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right)$.

By induction on m , using the Lemma



Proof of $\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right)$.

By induction on m , using the Lemma

- Case $m = 0$. Trivially from $q_0(x) = 1$ and $p_0(x) = x$.



Proof of $\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right)$.

By induction on m , using the Lemma

- Case $m = 0$. Trivially from $q_0(x) = 1$ and $p_0(x) = x$.
- $m \geq 1$. By induction on i , we prove that

$$\phi_m = \det(p_i \mathbf{I}_{m-i} - q_i \mathbf{A}_{m-i})$$

- $i = 0$: $\phi_m = \det(x \mathbf{I}_m - \mathbf{A}_m) = \det(p_0 \mathbf{I}_m - q_0 \mathbf{A}_m)$
- $i - 1 \Rightarrow i$: $\phi_m = \det(p_{i-1} \mathbf{I}_{m-i+1} - q_{i-1} \mathbf{A}_{m-i+1}) =$
 $= \det((p_{i-1}^2 - q_{i-1}^2) \mathbf{I}_{m-i} - p_{i-1} q_{i-1} \mathbf{A}_{m-i}) =$
 $= \det(p_i \mathbf{I}_{m-i} - q_i \mathbf{A}_{m-i})$



Proof of $\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right)$.

By induction on m , using the Lemma

- Case $m = 0$. Trivially from $q_0(x) = 1$ and $p_0(x) = x$.
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- $i = 0$: $\phi_m = \det(x \mathbf{I}_m - \mathbf{A}_m) = \det(p_0 \mathbf{I}_m - q_0 \mathbf{A}_m)$
- $i - 1 \Rightarrow i$: $\phi_m = \det(p_{i-1} \mathbf{I}_{m-i+1} - q_{i-1} \mathbf{A}_{m-i+1}) =$

$$\begin{aligned} &= \det((p_{i-1}^2 - q_{i-1}^2) \mathbf{I}_{m-i} - p_{i-1} q_{i-1} \mathbf{A}_{m-i}) = \\ &= \det(p_i \mathbf{I}_{m-i} - q_i \mathbf{A}_{m-i}) \end{aligned}$$

- The case $i = m$ gives

$$\begin{aligned} \phi_m(x) &= \det(p_m(x) \mathbf{I}_0 - q_m(x) \mathbf{A}_0) = \\ &= \det\left(q_m(x) \left(\frac{p_m(x)}{q_m(x)} \mathbf{I}_0 - \mathbf{A}_0\right)\right) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right) \end{aligned}$$



The spectrum of the binary hypertree $T_m = K_2^m$

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The spectrum of a generic two-term product $G_2 \square G_1$

4 Generalization of the hierarchical product

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Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

$$T_m = K_2^m$$

$$p_i = p_{i-1}^2 - q_{i-1}^2$$

$$q_i = p_{i-1}q_{i-1}$$

$$p_0 = x, \quad q_0 = 1$$

Corollary

- $\phi_{T_m}(x) = p_m(x)$
- $\phi_{T_m^*}(x) = q_m(x)$

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Proof.

$$G = K_1 \Rightarrow \phi_0(x) = x \Rightarrow \phi_{T_m}(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right) = p_m(x)$$

$$T_m^* = T_m - \mathbf{0} = \bigcup_{i=0}^{m-1} T_i \Rightarrow \phi_{T_m^*}(x) = \prod_{i=0}^{m-1} p_i(x) = q_m(x) \quad \square$$

Proposition

T_m , $m \geq 1$, has *distinct* eigenvalues $\lambda_0^m < \lambda_1^m < \dots < \lambda_{n-1}^m$, with $n = 2^m$, satisfying the following recurrence relation:

$$\lambda_{\frac{n}{2}+k}^m = \frac{\lambda_k^{m-1} + \sqrt{(\lambda_k^{m-1})^2 + 4}}{2}$$

$$\lambda_{n-k-1}^m = -\lambda_k^m$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \dots, n - 1$

The spectrum of the binary hypertree $T_m = K_2^m$

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Proof.

- $\lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}$, $\lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2}$
- T_m bipartite \Rightarrow its spectrum is symmetric with respect to 0
- $\text{sp } T_0 = \{0^1\} \Rightarrow$ the multiplicity of every λ_1^m is 1



The spectrum of the binary hypertree $T_m = K_2^m$ Properties of $\text{sp} T_m$

$$\lambda_i \in \text{sp} G \Rightarrow \lambda^2 - \lambda_i \lambda - 1 = 0$$

$$f_0(x) = \frac{x - \sqrt{x^2 + 4}}{2} \quad f_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}$$

$$m = 0 \Rightarrow \text{sp} T_0 = \{0\}$$

$$m = 1 \Rightarrow \lambda_0 = f_0(0) = -1, \lambda_1 = f_1(0) = 1$$

$$m = 2 \Rightarrow$$

$$\lambda_0 = f_0(-1) = f_0(f_0(0)) = -1.618 \quad \lambda_1 = f_0(1) = f_0(f_1(0)) = -0.618$$

$$\lambda_2 = f_1(-1) = f_1(f_0(0)) = 0.618 \quad \lambda_3 = f_1(1) = f_1(f_1(0)) = 1.618$$

...

$$m \text{ fixed, } \mathbf{i} = i_{m-1} \dots i_1 i_0 \in \mathbb{Z}_2^m \Rightarrow$$

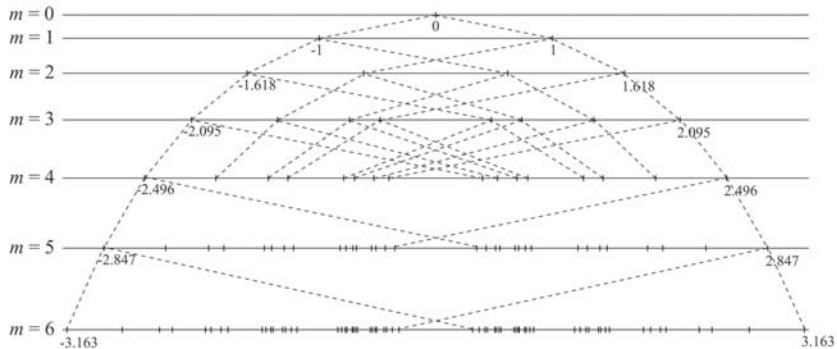
$$\Rightarrow \lambda_{\mathbf{i}} = (f_{i_{m-1}} \circ \dots \circ f_{i_1} \circ f_{i_0})(0)$$

The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

The distinct eigenvalues of the hypertree T_m for $0 \leq m \leq 6$.



The spectrum of the binary hypertree $T_m = K_2^m$

Proposition

The asymptotic behaviors of

- the spectral radius $\rho_k = \max_{0 \leq i \leq n-1} \{|\lambda_{\mathbf{i}}|\} = \lambda_{111\dots 1}$,
- the second largest eigenvalue $\theta_k = \lambda_{111\dots 10}$, and
- the minimum positive eigenvalue $\sigma_k = \min_{0 \leq i \leq n-1} \{|\lambda_{\mathbf{i}}|\} = \lambda_{100\dots 0}$

of the hypertree T_m are:

$$\rho_k \sim \sqrt{2k}, \quad \theta_k \sim \sqrt{2k}, \quad \sigma_k \sim 1/\sqrt{2k}$$

The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.



The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$



The spectrum of the binary hypertree $T_m = K_2^m$

Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$
- ρ_k and θ_k verify the recurrence
$$\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$$



Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

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$$\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$$

- Assuming $\lambda_k \sim \alpha k^\beta$

$$\begin{aligned} \alpha(k+1)^\beta &\sim \frac{\alpha k^\beta + \sqrt{\alpha^2 k^{2\beta} + 4}}{2} \Rightarrow \\ &\Rightarrow \alpha^2(k+1)^\beta [(k+1)^\beta - k^\beta] \sim 1 \end{aligned}$$

$$2(k+1)^{\frac{1}{2}} [(k+1)^{\frac{1}{2}} - k^{\frac{1}{2}}] = \frac{2(k+1)^{\frac{1}{2}}}{(k+1)^{\frac{1}{2}} + k^{\frac{1}{2}}} \rightarrow 1$$



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Theorem

Let G_1 and G_2 be two graphs on n_i vertices, with adjacency matrix \mathbf{A}_i and characteristic polynomial $\phi_i(x)$, $i = 1, 2$.

Consider the graph $G_1^* = G_1 - 0$, with adjacency matrix \mathbf{A}_1^* and characteristic polynomial ϕ_1^* .

Then the characteristic polynomial $\phi_H(x)$ of the hierarchical product $H = G_2 \square G_1$ is:

$$\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right)$$

The Hierarchical Product of Graphs

Algebraic properties

The spectrum of a generic two-term product $G_2 \square G_1$

$$\text{Proof of } \phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right).$$

Proof of $\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of H is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$\mathbf{A}_H = \mathbf{D}_1 \otimes \mathbf{A}_2 + \mathbf{A}_1 \otimes \mathbf{I}_2 = \begin{pmatrix} \mathbf{A}_2 & \mathbf{B} \\ \mathbf{B}^\top & \mathbf{A}_1^* \otimes \mathbf{I}_2 \end{pmatrix}$$

$$\text{where } \mathbf{B} = \begin{pmatrix} \mathbf{I}_2 & \dots \dots \dots & \overset{(\delta)}{\mathbf{I}_2} & \mathbf{0} & \mathbf{0} & \dots \dots \dots & \mathbf{0} \end{pmatrix}$$

Proof of $\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

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- The characteristic polynomial of H is

$$\phi_H(x) = \det(x\mathbf{I} - \mathbf{A}_H) = \det \begin{pmatrix} x\mathbf{I}_2 - \mathbf{A}_2 & -\mathbf{B} \\ -\mathbf{B}^\top & (x\mathbf{I}_1^* - \mathbf{A}_1^*) \otimes \mathbf{I}_2 \end{pmatrix}$$

Proof of $\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

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- Computing the determinant in $\mathbb{R}^{n_2 \times n_2}$:

$$\begin{aligned} \phi_H(x) &= \det([x\mathbf{I}_2 - \mathbf{A}_2]\phi_1^*(x)\mathbf{I}_2 + \phi_1(x)\mathbf{I}_2 - x\mathbf{I}_2\phi_1^*(x)) = \\ &= \det(\phi_1(x)\mathbf{I}_2 - \phi_1^*(x)\mathbf{A}_2) = \det \left(\phi_1^*(x) \left[\frac{\phi_1(x)}{\phi_1^*(x)} \mathbf{I}_2 - \mathbf{A}_2 \right] \right) = \\ &= \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right) \end{aligned}$$

Corollary

$$G_1 \text{ walk-regular} \Rightarrow \phi_H(x) = \left(\frac{\phi_1'(x)}{n_1} \right)^{n_2} \phi_2 \left(\frac{n_1 \phi_1(x)}{\phi_1'(x)} \right)$$

Proof.

$$\phi_1^*(x) = \frac{1}{n_1} \phi_1'(x)$$



Corollary

G graph of order $n_2 = N$ and characteristic polynomial $\phi_G \Rightarrow$ the characteristic polynomial of $H = G \square K_n$ is

$$\phi_H(x) = (x+1)^{N(n-2)} (x-n+2)^N \phi_G \left(\frac{(x+1)(x-n+1)}{(x-n+2)} \right)$$

Proof.

K_n is walk-regular, $\phi_{K_n} = (x-n+1)(x+1)^{n-1}$ and

$$\phi_{K_n}' = (x+1)^{n-1} + (n-1)(x-n+1)(x+1)^{n-2}$$



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Definition of the generalized hierarchical product

$$G_i = (V_i, E_i), \emptyset \neq U_i \subseteq V_i, i = 1, 2, \dots, N - 1$$

$H = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \dots \sqcap G_1(U_1)$ is the graph:

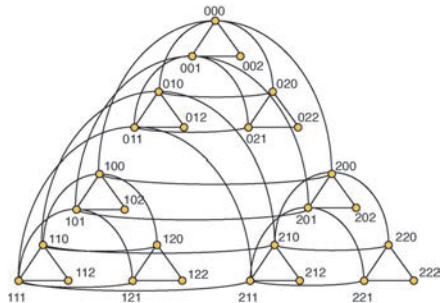
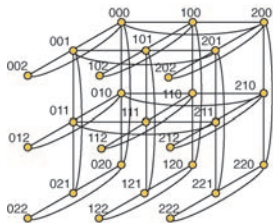
- vertices $V_N \times \dots \times V_2 \times V_1$
- if $x_j \sim y_j$ in G_j and $u_i \in U_i, i = 1, 2, \dots, j - 1$ then
 $x_N \dots x_{j+1} x_j u_{j-1} \dots u_1 \sim x_N \dots x_{j+1} y_j u_{j-1} \dots u_1$

Example

- For every $i, U_i = V_i \Rightarrow$
 $G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \dots \sqcap G_1(U_1) = G_N \square G_{N-1} \square \dots \square G_1$
- For every $i, U_i = \{0\} \Rightarrow$
 $G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \dots \sqcap G_1(U_1) = G_N \sqcap G_{N-1} \sqcap \dots \sqcap G_1$

Example

Two views of a generalized hierarchical product K_3^3 with $U_1 = U_2 = \{0, 1\}$.



Summary

- 1 Definition of the hierarchical product of graphs
- 2 Spectral properties
- 3 The particular case of T_m
- 4 Definition of the generalized hierarchical product

Further work

- 5 T_m , $\text{sp}T_m$ and $\bigcup_m \text{sp}T_m$ are structures with nice properties
- 6 Properties of the generalized hierarchical product

Thank you !!!